

MAXIMALLY SPARSE POLYNOMIALS HAVE SOLID AMOEBAS

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ABSTRACT. Let f be an ordinary polynomial in $\mathbb{C}[z_1, \dots, z_n]$ with no negative exponents and with no factor of the form $z_1^{\alpha_1} \dots z_n^{\alpha_n}$ where α_i are non zero natural integers. If we assume in addition that f is a maximally sparse polynomial (that its support is equal to the set of vertices of its Newton polytope), then a complement component of the amoeba \mathcal{A}_f in \mathbb{R}^n of the algebraic hypersurface $V_f \subset (\mathbb{C}^*)^n$ defined by f , has order lying in the support of f , which means that \mathcal{A}_f is solid. This gives an affirmative answer to Passare and Rullgård question in [PR2-01].

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1. INTRODUCTION

Mikael Passare and Hans Rullgård posed the following question:

" *Does every maximally sparse polynomial have a solid amoeba?* "

The purpose of this paper is to give an affirmative answer to this question. We use for this, Viro's patchworking principle applied to the Passare and Rullgård function (see Section 2 for definitions), Kapranov's theorem (see [K-00]) and some properties of complex tropical hypersurfaces. Note that here we can assume that f is a polynomial with no negative exponent and with no factor of the form z^α because our hypersurfaces lie in the algebraic torus $(\mathbb{C}^*)^n$.

A polynomial f is called *maximally sparse* if the support of f is equal to the set of the vertices of its Newton polytope Δ_f (see [PR2-01]), in other word, f is a polynomial with Newton polytope Δ with minimal number of monomials. An amoeba \mathcal{A} of degree Δ is called *solid* if the number of connected component of $\mathbb{R}^n \setminus \mathcal{A}$ is equal to the

number of vertices of Δ , which is the minimal number that an amoeba of degree Δ can have, (see [PR1-04] or [R-01]). We prove the following theorem for any $n \geq 1$:

Theorem 1.1. *Let V_f be an algebraic hypersurface in $(\mathbb{C}^*)^n$ defined by a maximally sparse polynomial f . Then the amoeba \mathcal{A}_f of V_f is solid.*

The paper is organized as follows. In section 2 we briefly review the definitions and the known results on tropical geometry and amoebas. We will then prove some properties of complex tropical hypersurfaces and we give a method for the construction of the set of arguments of a complex algebraic hypersurface defined by maximally sparse polynomial with Newton polytope a simplex in section 3. In section 4 we give the basic properties of Viro's local tropicalization. The proof of the main theorem will be given in section 5. It is based on tropical localization of a special deformation of a complex structure on a hypersurface to the so-called by Grigory Mikhalkin *complex tropical structure* which is the extrem possible degeneration. In Appendix B we give a geometric description of the set of arguments of the standard complex hyperplane, and finally in Appendix D we give an example which prove that maximally sparse polynomial is an optimal condition.

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2. PRELIMINARIES

In this paper we will consider only algebraic hypersurfaces V in the complex torus $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n \geq 1$ an integer. This means that V is the zero locus of a Laurent polynomial:

$$f(z) = \sum_{\alpha \in \text{supp}(f)} a_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} \quad (1)$$

where each a_α is a non-zero complex number and $\text{supp}(f)$ is a finite subset of \mathbb{Z}^n , called the support of the polynomial f , with convex hull, in \mathbb{R}^n , the Newton polytope Δ_f of f . Moreover we assume that $\text{supp}(f) \subset \mathbb{N}^n$ and f has no factor of the form z^α with $\alpha = (\alpha_1, \dots, \alpha_n)$.

The *amoeba* \mathcal{A}_f of an algebraic hypersurface $V_f \subset (\mathbb{C}^*)^n$ is by definition (see M. Gelfand, M.M. Kapranov and A.V. Zelevinsky [GKZ-94]) the image of V_f under the map :

$$\begin{aligned} \text{Log} : (\mathbb{C}^*)^n &\longrightarrow \mathbb{R}^n \\ (z_1, \dots, z_n) &\longmapsto (\log |z_1|, \dots, \log |z_n|). \end{aligned}$$

It was shown by M. Forsberg, M. Passare and A. Tsikh in [FPT-00] that there is an injective map between the set of components $\{E_\nu\}$ of $\mathbb{R}^n \setminus \mathcal{A}_f$ and $\mathbb{Z}^n \cap \Delta_f$:

$$\text{ord} : \{E_\nu\} \hookrightarrow \mathbb{Z}^n \cap \Delta_f \quad (2)$$

Theorem 2.1 (Foresberg-Passare-Tsikh, (2000)). *Each component of $\mathbb{R}^n \setminus \mathcal{A}_f$ is a convex domain and there exists a locally constant function:*

$$\text{ord} : \mathbb{R}^n \setminus \mathcal{A}_f \longrightarrow \mathbb{Z}^n \cap \Delta_f$$

which maps different components of the complement of \mathcal{A}_f to different lattice points of Δ_f .

Let \mathbb{K} be the field of the Puiseux series with real power, which is the field of the series $a(t) = \sum_{j \in A_a} \xi_j t^j$ with $\xi_j \in \mathbb{C}^*$ and $A_a \subset \mathbb{R}$ is well-ordered set with smallest element. It is well known that the field \mathbb{K} is algebraically closed and of characteristic zero, and it has a non-Archimedean valuation $\text{val}(a) = -\min A_a$:

$$\begin{cases} \text{val}(ab) &= \text{val}(a) + \text{val}(b) \\ \text{val}(a+b) &\leq \max\{\text{val}(a), \text{val}(b)\}, \end{cases}$$

and we put $\text{val}(0) = -\infty$. Let $f \in \mathbb{K}[z_1, \dots, z_n]$ be a polynomial as in (1) but the coefficients and the components of z are in \mathbb{K} , and $V_{\mathbb{K}}$ be the zero locus in $(\mathbb{K}^*)^n$ of the polynomial f . The following piecewise affine linear function $f_{trop} = \max_{\alpha \in \text{supp}(f)} \{\text{val}(a_\alpha) + \langle \alpha, x \rangle\}$ where \langle, \rangle is the scalar product in \mathbb{R}^n is called a *tropical polynomial*.

Definition 2.2. The tropical hypersurface Γ_f defined by the tropical polynomial f_{trop} is the subset of \mathbb{R}^n image under the valuation map of the algebraic hypersurface $V_{\mathbb{K}}$ over \mathbb{K} .

We have the following Kapranov's theorem (see [K-00]).

Theorem 2.3 (Kapranov, (2000)). *The tropical hypersurface Γ_f is the set of points in \mathbb{R}^n where the tropical polynomial f_{trop} is not smooth (called the corner locus of f_{trop}).*

Passare-Rullgård function.

Let A' be the subset of $\mathbb{Z}^n \cap \Delta_f$, image of $\{E_\nu\}$ under the order mapping (2). M. Passare and H. Rullgård proves in [PR1-04] that the spine Γ of the amoeba \mathcal{A}_f is given as a non-Archimedean amoeba defined by the tropical polynomial

$$f_{trop}(x) = \max_{\alpha \in A'} \{c_\alpha + \langle \alpha, x \rangle\},$$

where c_α is defined by:

$$c_\alpha = \text{Re} \left(\frac{1}{(2\pi i)^n} \int_{\text{Log}^{-1}(x)} \log \left| \frac{f(z)}{z^\alpha} \right| \frac{dz_1 \wedge \dots \wedge dz_n}{z_1 \dots z_n} \right) \quad (3)$$

where $x \in E_\alpha$, $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ and \langle, \rangle is the scalar product in \mathbb{R}^n . In other words, the spine of \mathcal{A}_f is defined as the set of points in \mathbb{R}^n where the piecewise affine linear function f_{trop} is not differentiable, or as the graph of this function where

\mathbb{R} is the semi-field $(\mathbb{R}; \max, +)$. Let us denote by τ the convex subdivision of Δ_f dual to the tropical variety Γ .

We define the Passare-Rullgård's function on the Newton polytope Δ_f as follows :

Let $\nu : \Delta_f \longrightarrow \mathbb{R}$ be the function such that :

- (i) if $\alpha \in \text{Vert}(\tau)$, then we set $\nu(\alpha) = -c_\alpha$
- (ii) if $\alpha \in \Delta_v \setminus \text{Vert}(\tau)$, where Δ_v is an element of the subdivision τ with maximal dimension, then we put $\nu(\alpha) = \langle \alpha, a_v \rangle + b_v$, where $y = \langle x, a_v \rangle + b_v$ is the equation of the hyperplane in $\mathbb{R}^n \times \mathbb{R}$ containing the points of coordinates $(\alpha, -c_\alpha) \in \mathbb{R}^n \times \mathbb{R}$ for $\alpha \in \text{Vert}(\Delta_v)$, $a_v = (a_{1,v}, \dots, a_{n,v}) \in \mathbb{R}^n$ and $b_v \in \mathbb{R}$.

If f is the polynomial given by (1), we define a family of polynomials $\{f_t\}_{t \in [0, \frac{1}{e}]}$ as follows :

$$f_t(z) = \sum_{\alpha \in \text{supp}(f)} \xi_\alpha t^{\nu(\alpha)} z^\alpha \quad (4)$$

where $\xi_\alpha = a_\alpha e^{\nu(\alpha)}$.

3. COMPLEX TROPICAL HYPERSURFACES.

Let h be a strictly positive real number and H_h be the self diffeomorphism of $(\mathbb{C}^*)^n$ defined by :

$$H_h : \begin{array}{ccc} (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ (z_1, \dots, z_n) & \longmapsto & (|z_1|^h \frac{z_1}{|z_1|}, \dots, |z_n|^h \frac{z_n}{|z_n|}), \end{array}$$

which defines a new complex structure on $(\mathbb{C}^*)^n$ denoted by $J_h = (dH_h)^{-1} \circ J \circ (dH_h)$ where J is the standard complex structure.

A J_h -holomorphic hypersurface V_h is a hypersurface holomorphic with respect to the J_h complex structure on $(\mathbb{C}^*)^n$. It is equivalent to say that $V_h = H_h(V)$ where $V \subset (\mathbb{C}^*)^n$ is an holomorphic hypersurface for the standard complex structure J on $(\mathbb{C}^*)^n$.

Recall that the Hausdorff distance between two closed subsets A, B of a metric space (E, d) is defined by:

$$d_{\mathcal{H}}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

Here we take $E = \mathbb{R}^n \times (S^1)^n$, with the distance defined as the product of the Euclidean metric on \mathbb{R}^n and the flat metric on $(S^1)^n$.

Definition 3.1. A complex tropical hypersurface $V_\infty \subset (\mathbb{C}^*)^n$ is the limit (with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$) of a sequence of a J_h -holomorphic hypersurfaces $V_h \subset (\mathbb{C}^*)^n$ when h tends to zero.

Using Kapranov's theorem [K-00], Mikhalkin gives an algebraic definition of a complex tropical hypersurfaces (see [M2-04]) as follows :

If $a \in \mathbb{K}^*$ is the Puiseux series $a = \sum_{j \in A_a} \xi_j t^j$ with $\xi \in \mathbb{C}^*$ and $A_a \subset \mathbb{R}$ is a well-ordered set with smallest element, then we have a non-Archimedean valuation on \mathbb{K} defined by $\text{val}(a) = -\min A_a$. We complexify the valuation map as follows :

$$\begin{aligned} w : \mathbb{K}^* &\longrightarrow \mathbb{C}^* \\ a &\longmapsto w(a) = e^{\text{val}(a) + i \arg(\xi_{-\text{val}(a)})} \end{aligned}$$

Let Arg be the argument map $\mathbb{K}^* \rightarrow S^1$ defined by: for any $a \in \mathbb{K}$ a Puiseux series so that $a = \sum_{j \in A_a} \xi_j t^j$, then $\text{Arg}(a) = e^{i \arg(\xi_{-\text{val}(a)})}$ (this map extends the map $\mathbb{C}^* \rightarrow S^1$ defined by $\rho e^{i\theta} \mapsto e^{i\theta}$ which we denote by Arg).

Applying this map coordinate-wise we obtain a map :

$$W : (\mathbb{K}^*)^n \longrightarrow (\mathbb{C}^*)^n$$

Theorem 3.2 (Mikhalkin, (2002)). *The set $V_\infty \subset (\mathbb{C}^*)^n$ is a complex tropical hypersurface if and only if there exists an algebraic hypersurface $V_{\mathbb{K}} \subset (\mathbb{K}^*)^n$ over \mathbb{K} such that $\overline{W(V_{\mathbb{K}})} = V_\infty$, where $\overline{W(V_{\mathbb{K}})}$ is the closure of $W(V_{\mathbb{K}})$ in $(\mathbb{C}^*)^n \approx \mathbb{R}^n \times (S^1)^n$ as a Riemannian manifold with metric defined by the standard Euclidean metric of \mathbb{R}^n and the standard flat metric of the torus.*

Recall that we have the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{K}^*)^n & \xrightarrow{W} & (\mathbb{C}^*)^n \\ & \searrow \text{Log}_{\mathbb{K}} \quad \swarrow \text{Log} & \\ & \mathbb{R}^n & \end{array}$$

where $\text{Log}_{\mathbb{K}}(z_1, \dots, z_n) = (\text{val}(z_1), \dots, \text{val}(z_n))$, which means that \mathbb{K} is equipped with the norm defined by $|z|_{\mathbb{K}} = e^{\text{val}(z)}$ for any $z \in \mathbb{K}^*$.

Let $V_\infty \subset (\mathbb{C}^*)^n$ be a complex tropical hypersurface of degree Δ . This means that $V_\infty = \overline{W(V_{\mathbb{K}})}$ where $V_{\mathbb{K}} \subset (\mathbb{K}^*)^n$ is an algebraic hypersurface over \mathbb{K} defined by the non-Archimedean polynomial $f_{\mathbb{K}}(z) = \sum_{\alpha \in \Delta \cap \mathbb{Z}^n} a_\alpha z^\alpha$. By Kapranov's theorem (see

[K-00]), $\Gamma = \text{Log}_{\mathbb{K}}(V_{\mathbb{K}})$ is a tropical hypersurface (called non-Archimedean amoeba associated to the polynomial $f_{\mathbb{K}}$ and denoted by $\mathcal{A}_{f_{\mathbb{K}}}$); we denote by τ the subdivision of Δ dual to Γ .

Definition 3.3. The complex numbers $w(a_\alpha)$ are called the complex tropical coefficients "defined" by V_∞ . They are well defined if we suppose that for some fixed index $\alpha_0 \in \Delta$, $w(a_{\alpha_0}) = 1$

In general we have the following (see Mikhalkin [M2-04] for $n = 2$):

Proposition 3.4. *Let $V_{\mathbb{K}} \subset (\mathbb{K}^*)^n$ as above. Then for any two indices α and β in $\text{Vert}(\tau)$, the quotients $\frac{w(a_\alpha)}{w(a_\beta)}$ are well defined and depend only on $W(V_{\mathbb{K}})$.*

Proof First of all, we may assume that α and β are adjacent to the same edge E of τ and we proceed by induction on vertices. Secondly, using an automorphism of $(\mathbb{C}^*)^n$ if necessary, we may assume that $E = [0, k] \times \{0\} \subset \mathbb{R}^n$. Let $E^* \subset \Gamma$ be the dual of E and $U \subset \mathbb{R}^n$ a small neighborhood of a point $x \in \text{Int}(E^*)$, then we have $\text{Log}^{-1}(U) \cap V_\infty = \text{Log}^{-1}(U) \cap V_{\infty, E}$ where $V_{\infty, E}$ is the complex tropical hypersurface defined in the same way of V_∞ but by taking the truncation of f to E . Indeed, the tropical monomials corresponding to lattice points in E dominate the tropical monomials corresponding to lattice points in $\Delta \setminus E$ (it's Kapranov's theorem [K-00]). Hence we can assume that $\Delta = E$ and prove the result for E .

Let $f_{\mathbb{K}}^E(z_1) = a_0 z_1^k + a_1 z_1^{k-1} + \dots + a_k \in \mathbb{K}[z_1]$ be a non-Archimedean polynomial in one variable such that $W(V_{f_{\mathbb{K}}^E}) = V_{\infty, E}$ (it can be seen as the truncation of $f_{\mathbb{K}}$ on E). The field \mathbb{K} is algebraically closed, hence the polynomial $f_{\mathbb{K}}^E$ has k roots r_1, \dots, r_k in

\mathbb{K} such that $\prod_{j=1}^k r_j = (-1)^k \frac{a_k}{a_0}$. On the other hand $V_{\infty, E}$ is the union of subsets $\cup_{i=1}^s \mathcal{C}_i$

in $(\mathbb{C}^*)^n$ defined by $\mathcal{C}_i = \{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n / \prod_{j=1}^{k_i} (z_1 - c_{ij}) = 0\}$ such that for any i we have $\log |c_{ij}| = c_i$ where c_i are constants depending only on $V_{\infty, E}$. Indeed, $\text{Log}(V_{\infty, E})$ is an hyperplane in \mathbb{R}^n orthogonal to the x_1 -axis, and $k = \sum k_i$. Any $w(r_j) \in \mathcal{C}_i$ for some i , so there exists j such that $w(r_j) = c_{ij}$. This means that $w(r_j)$ is a solution of the equation $z_1 - c_{ij} = 0$ (in the field of the complex numbers).

Then $\prod_{j=1}^k w(r_j) = \prod_i \prod_{j=1}^{k_i} c_{ij}$ and hence we have:

$$\frac{w(a_k)}{w(a_0)} = (-1)^k \prod_i \prod_{j=1}^{k_i} c_{ij}$$

which depends only on $V_{\infty, E}$ and hence only on V_∞ ; this proves Proposition 3.4. ■

Let $f_{\mathbb{K}}$ be a polynomial in $\mathbb{K}[z_1, \dots, z_n]$ with Newton polytope a simplex Δ such that $\text{supp}(f_{\mathbb{K}}) = \text{Vert}(\Delta)$; this implies that the corresponding non-Archimedean amoeba $\mathcal{A}_{f_{\mathbb{K}}}$ has only one vertex. Assume that there exists $\{g_{\mathbb{K}, u}\}_{u \in [0, 1]}$ a family of non-Archimedean polynomials defined by $g_{\mathbb{K}, u}(z) = f_{\mathbb{K}}(z) + \sum_{\beta \in A} a_{\beta, u} z^\beta$ where $A \subset (\Delta \cap \mathbb{Z}^n) \setminus \text{Vert}(\Delta)$ satisfy the following properties :

- (i) the complement components of the non-Archimedean amoeba $\mathcal{A}_{g_{\mathbb{K}, 1}}$ of $g_{\mathbb{K}, 1}$ are in bijection with $\text{Vert}(\Delta) \cup A$ by the order map and if we denote by τ the subdivision of Δ dual to the non-Archimedean amoeba $\mathcal{A}_{g_{\mathbb{K}, 1}}$ we assume that τ is a triangulation,
- (ii) let ν be the Passare-Rullgård function associated to the amoeba of $f_{\mathbb{K}}$ (in this case $\nu(\alpha) = -\log |a_\alpha|$ for any $\alpha \in \text{Vert}(\Delta)$), and for any $\beta \in A$ and

$0 \leq u \leq 1$, $\text{val}(a_{\beta,u}) = (1-u) \text{val}(a_{\beta,0}) + u \text{val}(a_{\beta,1})$, where $\text{val}(a_{\beta,0}) = -\nu(\beta)$, and $\arg(a_{\beta,u}) = \arg(a_{\beta,1})$ for each u such that $0 \leq u \leq 1$.

Let us denote by D_{std} the lift set in \mathbb{R}^n of the argument of the complex tropical hyperplane $W(H)$ where H is the hyperplane in $(\mathbb{K}^*)^n$ defined by the polynomial $z_1 + \dots + z_n + 1 = 0$, with degree the standard simplex $\Delta_{std} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_j \geq 0, x_1 + \dots + x_n \leq 1\}$.

Proposition 3.5. *Let $f_{\mathbb{K}}$ and $g_{\mathbb{K},u}$ having the above properties. Then $W(V_{f_{\mathbb{K}}}) = W(V_{g_{\mathbb{K},0}})$ if and only if A is empty.*

We can remark that if A is empty, $W(V_{f_{\mathbb{K}}}) = W(V_{g_{\mathbb{K},0}})$ because $g_{\mathbb{K},u} = f_{\mathbb{K}}$ in this case. Let Δ_i , for $1 \leq i \leq s$, be the simplices of dimension n of the triangulation τ . We have the following diagram:

$$(1) \quad \begin{array}{ccccc} & & & \mathbb{R}^n & \\ & & & \downarrow \pi & \\ (\mathbb{K}^*)^n & \xrightarrow{W} & (\mathbb{C}^*)^n & \xrightarrow{\text{Arg}} & (S^1)^n \\ & & \downarrow \text{Log} & & \\ & & \mathbb{R}^n & & \end{array}$$

where the map $\pi : \mathbb{R}^n \longrightarrow (S^1)^n$ is the projection of the universal covering of the torus, and $\text{Arg}(\rho e^{i\theta}) = e^{i\theta}$: see above.

Lemma 3.6. *Let $f_{\mathbb{K}}$ and $g_{\mathbb{K},u}$ with properties (i) and (ii). Then there exist invertible matrices $\{L_i\}_{i=1}^s \subset GL(n, \mathbb{R})$ with coefficients in \mathbb{Z} and positive determinant depending only on the triangulation τ (where s is the number of element of τ), and real vectors $\{(v_i)_{i=1,\dots,s} \mid v_i \in \mathbb{R}^n\}$ depending only on the complex tropical hypersurface $W(V_{g_{\mathbb{K},0}})$ such that :*

if v is the only vertex of the non-Archimedean amoeba $\mathcal{A}_{f_{\mathbb{K}}}$ we have

$\text{Arg}(\text{Log}^{-1}(v) \cap W(V_{g_{\mathbb{K},0}})) = \cup_{i=1}^s \mathcal{C}_i$ where $\mathcal{C}_i = (tr_{v_i} \circ^t L_i^{-1}(D_{std}) \setminus \mathcal{R}_i) / (2\pi\mathbb{Z})^n$ and tr_{v_i} are translations, with $\mathcal{R}_i / (2\pi\mathbb{Z})^n \subset \text{Arg}(W(V_{g_{\mathbb{K},1}}^{\Delta_i}))$ and depends only on the coefficients of $g_{\mathbb{K},1}$ with index in Δ_j 's which has a common face with Δ_i .

Proof We do not need the case $n = 1$ because Theorem 1.1 is obviously true for $n = 1$. However, we postpone the proof of Lemma 3.6. and Proposition 3.5. for $n = 1$ in the Appendix.

Case $n \geq 2$.

Let $f_{\mathbb{K}}(z) = a_0 + \sum_{j=1}^n a_j z_1^{\alpha_{1j}} \dots z_n^{\alpha_{nj}}$ and $V_{f_{\mathbb{K}}}$ its zero locus in $(\mathbb{K}^*)^n$. So $V_{f_{\mathbb{K}}}$ is the image of a hyperplane in $(\mathbb{K}^*)^n$ by the endomorphism $\tilde{L}_{\Delta} : (\mathbb{K}^*)^n \longrightarrow (\mathbb{K}^*)^n$ defined by the change of the variable $(z'_1, \dots, z'_n) \rightsquigarrow (z_1, \dots, z_n)$, $z'_j = z_1^{\alpha_{1j}} \dots z_n^{\alpha_{nj}}$ for $j = 1, \dots, n$.

This gives an endomorphism of the rings:

$$\begin{array}{ccc} \mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] & \longrightarrow & \mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \\ z_j' & \longmapsto & z_1^{\alpha_{1j}} \dots z_n^{\alpha_{nj}} \end{array}$$

Let ${}^tL_\Delta$ be the transpose matrix of the linear part L_Δ of the affine linear surjection which transform the standard simplex (i.e. with the $n+1$ vertices $(0, \dots, 0)$, $(1, 0, \dots, 0)$, $(0, \dots, 0, 1, 0, \dots, 0)$ and $(0, \dots, 0, 1)$) to Δ . So we obtain the automorphism of \mathbb{R}^n defined by:

$${}^tL_\Delta : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \begin{pmatrix} \text{val}(z_1) \\ \vdots \\ \text{val}(z_n) \end{pmatrix} & \longmapsto & \begin{pmatrix} \alpha_{11} & \dots & \alpha_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \dots & \alpha_{nn} \end{pmatrix} \begin{pmatrix} \text{val}(z_1) \\ \vdots \\ \text{val}(z_n) \end{pmatrix} = \begin{pmatrix} \text{val}(z_1') \\ \vdots \\ \text{val}(z_n') \end{pmatrix} \end{array}$$

and we have an homomorphism \tilde{L}_Δ of the multiplicative group $(\mathbb{K}^*)^n$ such that the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{K}^*)^n & \xleftarrow{\tilde{L}_\Delta} & (\mathbb{K}^*)^n \\ \text{Val} \downarrow & & \downarrow \text{Val} \\ \mathbb{R}^n & \xrightarrow{{}^tL_\Delta^{-1}} & \mathbb{R}^n. \end{array}$$

Then the image of the hyperplane in $(\mathbb{K}^*)^n$ defined by the polynomial $f_{\mathbb{K}, \text{td}}(z') = a_0 + \sum_{j=1}^n a_j z_j'$ is precisely the hypersurface defined by $f_{\mathbb{K}}$. Here we have

$$L_\Delta = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix}.$$

So we obtain the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{K}^*)^n & \xleftarrow{\tilde{L}_\Delta} & (\mathbb{K}^*)^n \\ \tilde{W} \downarrow & & \downarrow \tilde{W} \\ \mathbb{C}^n & \xrightarrow{\bar{L}_\Delta} & \mathbb{C}^n \\ \exp \downarrow & & \downarrow \exp \\ (\mathbb{C}^*)^n & \xrightarrow{M_\Delta} & (\mathbb{C}^*)^n \end{array}$$

where $\tilde{W} = (\text{Val}, \text{Arg})$ (i.e. $\tilde{W}(z) = (\text{val}(z) + i \arg(z))$), and M_Δ is the endomorphism of $(\mathbb{C}^*)^n$ covered by $\bar{L}_\Delta = ({}^tL_\Delta)^{-1} \otimes \mathbb{C}$. Using the map $W = \exp \circ \tilde{W}$, we obtain the

following commutative diagram:

$$\begin{array}{ccc}
 (\mathbb{K}^*)^n & \xleftarrow{\tilde{L}_\Delta} & (\mathbb{K}^*)^n \\
 W \downarrow & & \downarrow W \\
 (\mathbb{C}^*)^n & \xrightarrow{M_\Delta} & (\mathbb{C}^*)^n \\
 \text{Log} \downarrow & & \downarrow \text{Log} \\
 \mathbb{R}^n & \xrightarrow{{}^t L_\Delta^{-1}} & \mathbb{R}^n
 \end{array}$$

The degree of the map M_Δ is equal to the determinant of L_Δ i.e. $\deg(M_\Delta) = \det(L_\Delta) = n! \text{Vol}(\Delta)$. Let H_a be the hyperplane in $(\mathbb{K}^*)^n$ defined by the polynomial $f_{\mathbb{K}, \text{std}}(z') = a_0 + \sum_{j=1}^n a_j z'_j$, then $W(V_{f_{\mathbb{K}}}) = M_\Delta(W(H_a))$.

Claim 1: Let $f_{\mathbb{K}, \text{std}}(z) = 1 + \sum_{j=1}^n z_j$. Then we have $W(H_a) = \tau_v \circ W(H)$ where τ_v denotes the translation in the multiplicative algebraic torus $(\mathbb{C}^*)^n$ by an element $v \in (\mathbb{C}^*)^n$ well defined by the coefficients of $f_{\mathbb{K}, \text{std}}$.

Proof We are in the algebraic torus, then we can assume that $a_0 = 1$ and the valuation of each other coefficients is zero. Indeed, let Φ_a be the automorphism of $(\mathbb{K}^*)^n$ defined by $\Phi_a(z_1, \dots, z_n) = (t^{\text{val}(a_1)} z_1, \dots, t^{\text{val}(a_n)} z_n)$, then $f_{\mathbb{K}, \text{std}} \circ \Phi_a$ has the required assertion and $\text{Arg}(W(H_a)) = \text{Arg}(W(V_{f_{\mathbb{K}, \text{std}} \circ \Phi_a}))$. We can see that if $f_{\mathbb{K}, \text{std}} \circ \Phi_a(z) = 1 + \sum_{i=1}^n a'_i z_i$ then $\text{Arg}(W(V_{f_{\mathbb{K}, \text{std}} \circ \Phi_a})) = \tau_v(\text{Arg}(W(H)))$ where τ_v is the multiplication in the real torus $(S^1)^n$ by $(e^{i \arg(a'_1)}, \dots, e^{i \arg(a'_n)})$. ■

Let $\tilde{v} = {}^t L_\Delta^{-1}(v)$ (where v is viewed as a vector in the universal covering), then we have $W(V_{f_{\mathbb{K}}}) = M_\Delta(W(H_a)) = \tau_{\tilde{v}} \circ M_\Delta(W(H))$; so we obtain $\text{Arg}(W(V_{f_{\mathbb{K}}})) = (\tau_{\tilde{v}} \circ {}^t L_\Delta^{-1}(D_{\text{std}})) / (2\pi\mathbb{Z})^n$ where $D_{\text{std}} \subset \mathbb{R}^n$ is the lift set of the argument of $W(H)$ in the universal covering of $(S^1)^n$ and $\tilde{v} = {}^t L_\Delta^{-1}(\arg(a'_1), \dots, \arg(a'_n)) \in \mathbb{R}^n$.

We can remark that for any $\Delta_i \in \tau$, the argument of $W(V_{g_{\mathbb{K}, u}^{\Delta_i}})$ is independent of u , because the deformation is given such that the combinatorial type of the tropical hyperplane $\text{Log} \circ W(V_{g_{\mathbb{K}, u}^{\Delta_i}})$ is the same for any u and the argument of the coefficients of $g_{\mathbb{K}, u}^{\Delta_i}$ are independent of u by construction. We denote by $ad(i)$ the set of j so that Δ_j is adjacent to Δ_i . Let \mathcal{R}_{ij} be the subset of the lift in \mathbb{R}^n of $\text{Arg}(W(V_{g_{\mathbb{K}, 1}^{\Delta_i}})) \cap \text{Arg}(W(V_{g_{\mathbb{K}, 1}^{\Delta_j}}))$ not in the lift of $\text{Arg}(W(V_{g_{\mathbb{K}, 1}^{\Delta_i \cup \Delta_j}}))$, and put $\mathcal{R}_i = \cup_{j \in ad(i)} \mathcal{R}_{ij}$. The \mathcal{R}_{ij} depends only on the coefficients of $g_{\mathbb{K}, 1}$ with index in $\Delta_i \cup \Delta_j$. Hence, at the limit (i.e. $u = 0$), we have:

$$\begin{aligned}
\text{Arg}(W(V_{g_{\mathbb{K},0}})) &= \bigcup_{\Delta_i \in \tau} \{\text{Arg}((W(V_{g_{\mathbb{K},1}^{\Delta_i}})) \setminus \mathcal{R}_i)/(2\pi\mathbb{Z})^n\} \\
&= \bigcup_{\Delta_i \in \tau} (\tau_{\tilde{v}_i} \circ {}^tL_{\Delta_i}^{-1}(D_{std}) \setminus \mathcal{R}_i)/(2\pi\mathbb{Z})^n
\end{aligned}$$

where L_{Δ_i} is the linear part of the affine linear surjection map between the standard simplex and Δ_i , \tilde{v}_i are the translation vectors on the torus corresponding to the truncations $g_{\mathbb{K},1}^{\Delta_i}$ as described above. \blacksquare

Proof of Proposition 3.5. For each u , let us take the following notations : $V_{\infty,u} = W(V_{g_{\mathbb{K},u}})$ and $V_{\infty,f} = W(V_{f_{\mathbb{K}}})$. The principal arguments of Proposition 3.5, are the fact that, firstly the non-Archimedean amoeba Γ_{∞} has only one vertex. Secondly, if $f_{\mathbb{K}}$ is maximally sparse, then the lifting of the boundary $\overline{\partial \text{Arg}(V_{\infty,f})}$ of the closure of the set of argument of the complex tropical hypersurface $V_{\infty,f}$ (called by M. Passare the coamoeba of the complex tropical hypersurface $V_{\infty,f}$ and denoted by $\text{co}\mathcal{A}_{V_{\infty,f}}$), are the hyperplanes orthogonal to the edges $E_{\alpha_i\alpha_j}$ of Δ ; in addition $\overline{\text{Arg}(V_{\infty,0})}$ contains extra-pieces, where $V_{\infty,0} = \lim_{u \rightarrow 0} V_{\infty,u}$. Indeed, let H_{ij} be the hyperplane image under ${}^tL^{-1}$ of the hyperplane H_{ij}^{std} orthogonal to the edge $E_{\alpha_i\alpha_j}^{std}$ of the standard simplex such that $E_{\alpha_i\alpha_j} = L(E_{\alpha_i\alpha_j}^{std})$. Then we have $\langle {}^tL_{\Delta_i}^{-1}(H_{ij}^{std}), E_{\alpha_i\alpha_j} \rangle = \langle H_{ij}^{std}, L^{-1}(E_{\alpha_i\alpha_j}) \rangle = \langle H_{ij}^{std}, E_{\alpha_i\alpha_j}^{std} \rangle = 0$. Secondly, each edge $E_{\alpha_i\alpha_j}$ of Δ is dual to an $(n-1)$ -polyhedron $E_{ij}^* \subset \Gamma_{\infty} = \mathcal{A}_{f_{\mathbb{K}}} \subset \mathbb{R}^n$. Let $x \in E_{ij}^*$ and U be a small ball in \mathbb{R}^n centered at x . Then, using Kapranov's theorem [K-00], we have:

$$\overline{\text{Arg}(\text{Log}^{-1}(U) \cap V_{\infty,f})} \subset \mathcal{N}_{\varepsilon}(\text{Arg}\{z \in (\mathbb{C}^*)^n / a_{\alpha_i}z^{\alpha_i} + a_{\alpha_j}z^{\alpha_j} = 0\}),$$

where α_i and α_j are the vertices of Δ bounding the edge $E_{\alpha_i\alpha_j}$ and $\mathcal{N}_{\varepsilon}$ designate the ε -neighborhood. Hence we obtain:

$$\arg(a_{\alpha_i}) + \langle \alpha_i, \text{Arg}(z) \rangle = \pi + \arg(a_{\alpha_j}) + \langle \alpha_j, \text{Arg}(z) \rangle + 2k\pi,$$

where $k \in \mathbb{Z}$, $\text{Arg}(z) = (\arg(z_1), \dots, \arg(z_n))$, $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in})$, $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn})$, and \langle, \rangle is the Euclidean scalar product. So the hyperplanes H_{ij} in \mathbb{R}^n of equations:

$$\arg(a_{\alpha_i}) - \arg(a_{\alpha_j}) + \sum_{l=1}^n (\alpha_{il} - \alpha_{jl}) \cdot x_l = (2k+1)\pi,$$

are the boundary of the set $\overline{\text{Arg}(V_{\infty,f})}$ because the set of arguments of $V_{\infty,f}$ can be only in one side of H_{ij} . Let us describe now the boundary of $\overline{\text{Arg}(V_{\infty,0})}$.

Lemma 3.7. *For any $u \geq 0$, there are extra-pieces $\mathcal{P}_{j,u}$ contained in $\text{Arg}(V_{\infty,u})$ with no vanishing volume, such that $\mathcal{P}_{j,u} \cap \text{Arg}(V_{\infty,f}) = \emptyset$.*

Proof Let us call the hyperplanes H_{ij} *external hyperplane*, and assume that Δ contains just one point β in the interior of Δ . Let $V_{\infty,g_u^{\Delta_i}}$ for $i = 0, \dots, n-1$

be the complex tropical hypersurface image under the map W of the hypersurface in $(\mathbb{K}^*)^n$ defined by the truncation of $g_{\mathbb{K},u}$ to Δ_i where Δ_i is an element of the triangulation $\tau = \{\Delta_0, \dots, \Delta_n\}$. There exists an external hyperplane $H_{rs} \subset \overline{\partial \text{Arg}(V_{\infty,f})}$ intersecting the union $\cup_{i=0}^n \text{Arg}(V_{\infty,g_u^{\Delta_i}})$ in its interior. Because otherwise, this means that each connected component of the complement of $\text{Arg}(V_{\infty,f})$ in the torus $(S^1)^n$ is strictly contained in some complement component of $\text{Arg}(V_{\infty,g_u^{\Delta_i}})$ (indeed, if that inclusion is not strict, then there is at least one face of some complement component of $\text{Arg}(V_{\infty,f})$ intersecting the interior of $\text{Arg}(V_{\infty,g_u^{\Delta_i}})$), and then $\text{Vol}((S^1)^n \setminus \text{Arg}(V_{\infty,g_u^{\Delta_i}})) > \text{Vol}((S^1)^n \setminus \text{Arg}(V_{\infty,f}))$; that contradicts the fact that the volume of the last two sets is the same (see Appendix B).

Hence $\text{Arg}(V_{\infty,u})$ contains some extra-pieces $\mathcal{P}_{j,u}$ in the exterior of $\text{Arg}(V_{\infty,f})$, because the set of argument of $V_{\infty,f}$ can be only in one side of the external hyperplanes. Let $\mathcal{P}_j = \lim_{u \rightarrow 0} \mathcal{P}_{j,u}$, then $\text{Vol}(\mathcal{P}_j) \neq 0$. Indeed, if $\text{Vol}(\mathcal{P}_j) = 0$, this means that the valuation of the coefficient with index β tends to $-\infty$ (in other word, this means that the coefficient with index β tends to zero), which is not the case by construction, because the valuation of that coefficient tends to $-\nu_{PR}(\beta)$, which is finite. \blacksquare

On the other hand we have: $\text{Arg}(V_{\infty,0}) = \lim_{u \rightarrow 0} \text{Arg}(V_{\infty,u})$. So the boundary of $\overline{\text{Arg}(V_{\infty,0})}$ contains other pieces in $\partial \mathcal{P}_j$, not in the hyperplanes H_{ij} . Hence $\text{Arg}(V_{\infty,f})$ cannot be equal to $\text{Arg}(V_{\infty,0})$.

If A contains more than one point, we use induction on the cardinality of A , and we subdivide Δ into at most $n+1$ simplex with common vertex $\beta \in A$. Using the same reasoning we have the result.

Remark 3.8. For any u , there are extra-pieces $\mathcal{P}_{j,u}$ with no vanishing volume in $\text{Arg}(V_{\infty,u})$ (see for example figure 2 and 3 for $n = 2$), corresponding to the dual of the edges of the subdivision of Δ (dual to $\Gamma_{\infty,u}$) other than the edges of Δ . So the sets $\text{Arg}(V_{\infty,0})$ and $\text{Arg}(V_{\infty,f})$ cannot be equal even if u tends to some negative real number, this means even if the valuation of the coefficients a_β 's are above the hyperplane in \mathbb{R}^{n+1} passing through the points of coordinates $(\alpha, \nu(\alpha))$ with α in $\text{Vert}(\Delta)$. If we add in the hypothesis of the Proposition 3.5 that the non-Archimedean amoeba $\mathcal{A}_{f_{\mathbb{K}}}$ has only one vertex, then Proposition 3.5 and Lemma 3.7 are true, even if Δ is not a simplex.

Remark 3.9. (i) The number of connected component of $\text{Arg}(W(V_{f_{\mathbb{K}},std}))$, when we remove the real points, is $2^n - 2$ and the volume of any component is $\frac{n-1}{n}\pi^n$,
(ii) if we denote by $\tilde{\mathcal{P}}_l$ the lift set in \mathbb{R}^n of \mathcal{P}_l , then any component of $\tilde{\mathcal{P}}_l$ is a polyhedron (triangle for $n = 2$ and not convex for $n > 2$) with vertices in $\{(k_1\pi, \dots, k_n\pi)\}_{k_i \in \mathbb{Z}}$,
(iii) if we assume that u can have negative values then we have:

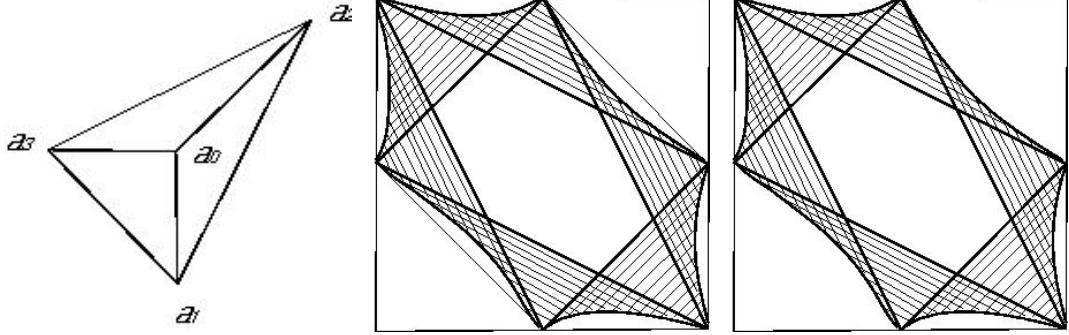


FIGURE 1. Example of extra-pieces in dimension two for some choice of the coefficients, and the adjacent element of the subdivision

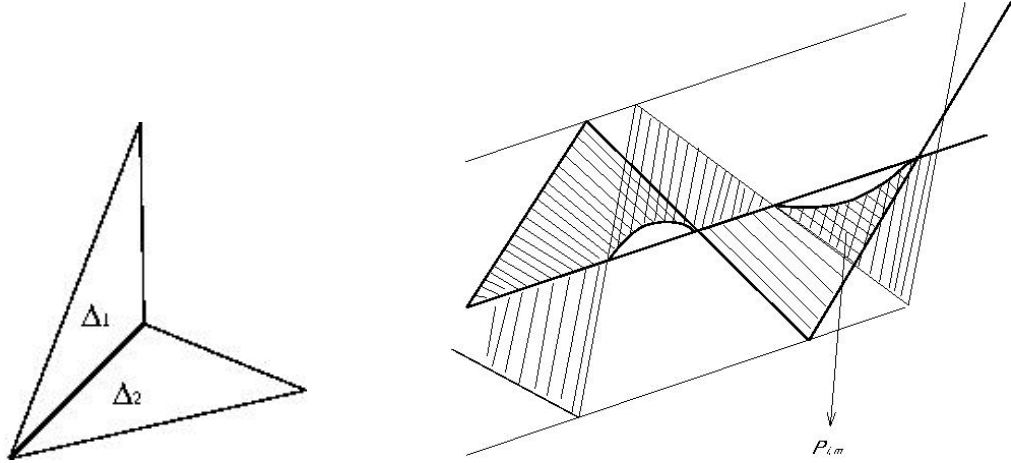


FIGURE 2. Example of extra-pieces in dimension two, and the adjacent element of the subdivision

- (1) if $0 < u \leq 1$ then we can choose the coefficients such that the argument of $W(V_{g_{\mathbb{K}}, u})$ is constant and the tropical hypersurface $\text{Log} \circ W(V_{g_{\mathbb{K}}, u})$ vary,
- (2) if $u < 0$ then the argument of $W(V_{g_{\mathbb{K}}, u})$ varies and the tropical hypersurface $\text{Log} \circ W(V_{g_{\mathbb{K}}, u})$ is constant,
- (3) the set of arguments of $W(V_{f_{\mathbb{K}}})$ is called the coamoeba of the complex tropical hypersurface $W(V_{f_{\mathbb{K}}})$ and is in the same time the limit of the coamoebas of some sequence of J_t -holomorphic hypersurfaces. We describe the two last points with more details in the forthcoming papers [N1-07] and [N2-07].

We draw in figure 4 the set of argument (known as the coamoeba, for more detail see [N1-07]) of the curve in $(\mathbb{C}^*)^2$ defined by the polynomial $f_1(z, w) = w^3 z^2 + w z^3 + 1$

where the matrix ${}^tL_1^{-1}$ is equal to $\frac{1}{7} \begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}$ and in figure 5 the coamoeba of the curve in $(\mathbb{C}^*)^2$ defined by the polynomial $f_2(z, w) = w^2z^2 + z + w$ where the matrix ${}^tL_2^{-1}$ is equal to $\frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$

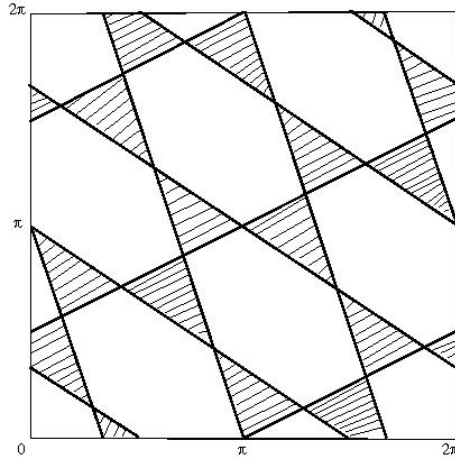


FIGURE 3. The image of the curve defined by the polynomial $f_1(z, w) = wz^3 + z^2w^3 + 1$ under the argument map Arg

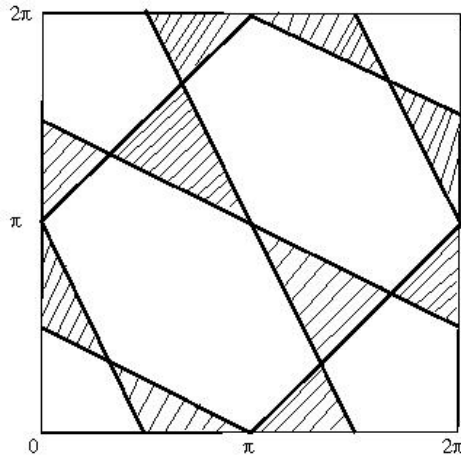


FIGURE 4. The image of the curve defined by the polynomial $f(z, w) = z + w + z^2w^2$ under the argument map Arg

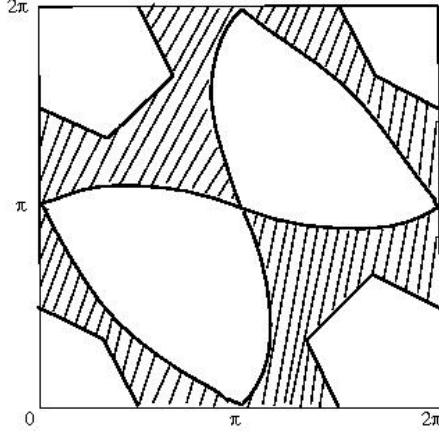


FIGURE 5. The image of the curve defined by the polynomial $f(z, w) = z + w + \frac{zw}{2} + z^2w^2$ under the argument map Arg

4. VIRO'S PATCHWORKING PRINCIPLE.

Let Δ be a convex integer polytope and $\tau = \cup_{v=1}^l \Delta_v$ a convex integer subdivision of Δ (we can see Viro's theory in [V-90] for more details of this definition and generally on the patchwork principle). This means that there exists a convex piecewise affine linear map $\nu : \Delta \rightarrow \mathbb{R}$ so that:

- (i) $\nu|_{\Delta_v}$ is affine linear for each v ,
- (ii) if $\nu|_U$ is affine linear for some open set $U \subset \Delta$, then there exists v such that $U \subset \Delta_v$.

Let $\tilde{\Delta}$ be the extended polyhedral of Δ associated to ν , that is the convex hull of the set $\{(\alpha, u) \in \Delta \times \mathbb{R} \mid u \geq \nu(\alpha)\}$. For any $\Delta_v \in \tau$, let $\lambda(x) = \langle x, a_v \rangle + b_v$ be the affine linear map defined on Δ such that $\lambda|_{\Delta_v} = \nu|_{\Delta_v}$ where \langle, \rangle is the scalar product in \mathbb{R}^n , $a_v = (a_{v,1}, \dots, a_{v,n}) \in \mathbb{R}^n$ and b_v is a real number. We put $\nu' = \nu - \lambda$ and we define the family of polynomials $\{f'_t\}_{t \in [0, \frac{1}{e}]}$ by:

$$f'_t(z) = \sum_{\alpha \in A} \xi_\alpha t^{\nu'(\alpha)} z^\alpha$$

where $\xi_\alpha \in \mathbb{C}$. Then we have:

$$\begin{aligned} f'_t(z) &= t^{-b_v} \sum_{\alpha \in A} \xi_\alpha t^{\nu(\alpha)} (z_1 t^{-a_{v,1}})^{\alpha_1} \dots (z_n t^{-a_{v,n}})^{\alpha_n} \\ &= t^{-b_v} f_t \circ \Phi_{\Delta_v, t}^{-1}(z) \end{aligned}$$

where f_t is the polynomial defined by:

$$f_t(z) = \sum_{\alpha \in A} \xi_\alpha t^{\nu(\alpha)} z^\alpha$$

and $\Phi_{\Delta_v, t}$ is the self diffeomorphism of $(\mathbb{C}^*)^n$ defined by:

$$\begin{aligned} \Phi_{\Delta_v, t} : (\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n \\ (z_1, \dots, z_n) &\longmapsto (z_1 t^{a_{v,1}}, \dots, z_n t^{a_{v,n}}). \end{aligned}$$

This means that the polynomials f'_t and $f_t \circ \Phi_{\Delta_v, t}^{-1}$ defines the same hypersurface. So we have:

$$V_{f'_t} = V_{f_t \circ \Phi_{\Delta_v, t}^{-1}} = \Phi_{\Delta_v, t}(V_{f_t})$$

Let Γ_t be the spine of the amoeba $\mathcal{A}_{H_t(V_{f_t})}$ where H_t denotes the self diffeomorphism of $(\mathbb{C}^*)^n$ defined by H_h with $h = -\frac{1}{\log t}$ and $\text{Log}_t = \text{Log} \circ H_t$. Let $U(v)$ be a small ball in \mathbb{R}^n with center the vertex of Γ_t dual to Δ_v , $f_t^{\Delta_v}$ be the truncation of f_t to Δ_v , and V_{∞, Δ_v} is the complex tropical hypersurface with tropical coefficients of index $\alpha \in \Delta_v$ (i.e., $V_{\infty, \Delta_v} = \lim_{t \rightarrow 0} H_t(V_{f_t^{\Delta_v}})$). Using Kapranov's theorem (see [K-00]), we obtain the following proposition (called tropical localization by Mikhalkin, see [M2-04]):

Proposition 4.1. *For any $\varepsilon > 0$ there exist t_0 such that if $t \leq t_0$ then the image under $\Phi_{\Delta_v, t} \circ H_t^{-1}$ of $H_t(V_{f_t}) \cap \text{Log}^{-1}(U(v))$ is contained in the ε -neighborhood of the image under $\Phi_{\Delta_v, t} \circ H_t^{-1}$ of the complex tropical hypersurface V_{∞, Δ_v} , with respect to the product metric in $(\mathbb{C}^*)^n \approx \mathbb{R}^n \times (S^1)^n$.*

Proof By decomposition of f'_t , we have:

$$f'_t(z) = t^{-b_v} \sum_{\alpha \in \Delta_v \cap A} \xi_\alpha t^{\nu(\alpha) - \langle \alpha, a_v \rangle} z^\alpha + \sum_{\alpha \in A \setminus \Delta_v} \xi_\alpha t^{\nu(\alpha) - \langle \alpha, a_v \rangle - b_v} z^\alpha \quad (5)$$

On the other hand we have the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\Phi_{\Delta_v, t}} & (\mathbb{C}^*)^n \\ \text{Log}_t \downarrow & & \downarrow \text{Log}_t \\ \mathbb{R}^n & \xrightarrow{\phi_{\Delta_v}} & \mathbb{R}^n \end{array} \quad (6)$$

such that if $v = (a_{v,1}, \dots, a_{v,n}) \in \mathbb{R}^n$ is the vertex of the tropical hypersurface Γ dual to the element Δ_v of the subdivision τ , then $\phi_{\Delta_v}(x_1, \dots, x_n) = (x_1 - a_{v,1}, \dots, x_n - a_{v,n})$. Let $U(v)$ be a small open ball in \mathbb{R}^n centered at v .

Assume that $\text{Log}_t(z) \in \phi_{\Delta_v}(U(v))$ and z is no singular in V_{f_t} . Then the second sum in (5) converges to zero when t tends to zero, because by the choice of z and $U(v)$, the tropical monomials in $f'_{trop, t}$ corresponding to lattice points of Δ_v dominates the monomials corresponding to lattice points of $A \setminus \Delta_v$. But the first sum in (5) is just a polynomial defining the hypersurface $\Phi_{\Delta_v, t}(V_{f_t^{\Delta_v}})$.

By the commutativity of the last diagram, if we take $z \in V_{f'_t}$ such that $\text{Log}_t(z) \in \phi_{\Delta_v}(U(v))$ then $\text{Log}_t \circ \Phi_{\Delta_v, t}^{-1}(z) \in U(v)$ and hence $H_t(\Phi_{\Delta_v, t}^{-1}(z)) \in \text{Log}^{-1}(U(v))$. So the image under $\Phi_{\Delta_v, t} \circ H_t^{-1}$ of $H_t(V_{f_t}) \cap \text{Log}^{-1}(U(v))$ is contained in an ε -neighborhood of the image under $\Phi_{\Delta_v, t} \circ H_t^{-1}$ of $H_t(V_{f_t^{\Delta_v}})$ for sufficiently small t and the proposition is done because V_{∞, Δ_v} is the limit when t tends to zero of the sequence of J_t -holomorphic

hypersurfaces $H_t(V_{f_t^{\Delta_v}})$ (by taking a discrete sequence t_k converging to zero if necessary).

■

5. MAXIMALLY SPARSE POLYNOMIALS AND PROOF OF THE MAIN THEOREM

From now we assume that the polynomial f is maximally sparse i.e. $\text{supp}(f) = \text{Vert}(\Delta_f)$. The family of polynomials (4) can be considered as polynomial $f_{\mathbb{K}}$ with coefficients in the non-Archimedean field \mathbb{K} of Puiseux series with coefficients in \mathbb{C} . So if we denote by $V_{\mathbb{K}}$ the hypersurface in $(\mathbb{K}^*)^n$ defined by the polynomial $f_{\mathbb{K}}$ and $-1/\log t$ the contraction of $\mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $(x_1, \dots, x_n) \mapsto (-\frac{x_1}{\log t}, \dots, -\frac{x_n}{\log t})$ for $t \in]0, \frac{1}{e}]$, and V_{f_t} the hypersurface defined by the complex polynomial f_t , then we have the following theorem of M. Passare and H. Rullgård in [PR1-04] and G. Mikhalkin in [M1-02]:

Theorem 5.1. *The non-Archimedean amoeba $\mathcal{A}_{V_{\mathbb{K}}} = \Gamma_{\infty} \subset \mathbb{R}^n$ of the hypersurface $V_{\mathbb{K}} \subset (\mathbb{K}^*)^n$ is the limit (with respect to the Hausdorff metric on compacts) of $(-1/\log t)(\mathcal{A}_{V_{f_t}})$ when t tends to zero.*

On one hand the non-Archimedean amoeba Γ_{∞} is the variety of the tropical polynomial $\max_{\alpha \in \text{supp}(f)} \{-\nu(\alpha) + \langle \alpha, x \rangle\}$ and on the other hand the limit of $(-1/\log t)(\mathcal{A}_{V_{f_t}})$ is the limit of the spines Γ_t of the amoebas $\mathcal{A}_{H_t(V_{f_t})}$ of the J_t -holomorphic hypersurface $H_t(V_{f_t})$. Hence Γ_{∞} is solid (because $\text{supp}(f) = \text{Vert}(\Delta_f)$ and any vertex of Δ_f corresponds to a complement component of the amoeba, see [PR1-04]) and the subdivision τ_{∞} of Δ_f dual to Γ_{∞} has the following properties : by a small perturbation of the coefficient vector of f if necessary, we can assume that the subdivision τ_{∞} is a triangulation (this means that each element of τ_{∞} is a simplex). Let $V_{f_{\vec{a}}}$ be the hypersurface defined by the coefficient vector $a = (a_1, \dots, a_r)$, then by the lower semi-continuity of the function $a \mapsto \#\{\text{component of } \mathbb{R}^n \setminus \mathcal{A}_{f_{\vec{a}}}\}$, if the coefficient vector \vec{b} is close enough to \vec{a} , then the number of complement components of $\mathbb{R}^n \setminus \mathcal{A}_{f_{\vec{b}}}$ is greater or equal to the number of complement components of $\mathbb{R}^n \setminus \mathcal{A}_{f_{\vec{a}}}$ (see [FPT-00]). Hence if we prove that $\mathcal{A}_{f_{\vec{b}}}$ is solid then $\mathcal{A}_{f_{\vec{a}}}$ is solid too. So we can suppose for our problem that τ_{∞} is a triangulation.

Note that the vertices of any simplex of τ_{∞} are contained in $\text{Vert}(\Delta_f)$. Let $\mathcal{L} \subset \Delta_f \cap \mathbb{Z}^n$ be the complement of $\text{Vert}(\Delta_f)$ in the image of the order mapping i.e. $\mathcal{L} = \{\alpha \in \Delta_f \cap \mathbb{Z}^n \mid E_{\alpha}^c \neq \emptyset \text{ and } a_{\alpha} = 0\}$. Using the triangulation τ_{V_f} dual to the spine of the amoeba \mathcal{A}_{V_f} , we define a triangulation $\tau_{V_f, \mathcal{L}}$ of Δ_f satisfying the following properties (see Appendix. C for more details and notations) :

- (i) $\tau_{V_f, \mathcal{L}} = \tau_{\infty}$,
- (ii) there is a deformation $\{H_t(V_{f_t})\}_{t \in]0, \frac{1}{e}]}$ of the hypersurface V_f so that for any $t \in]0, \frac{1}{e}]$ the spines Γ_t of the amoebas $\mathcal{A}_{H_t(V_{f_t})}$ have the same combinatorial

type than Γ_f , the spine of the amoeba of the initial hypersurface V_f ; and $\lim_{t \rightarrow 0} \Gamma_t = \Gamma_\infty$. This means that the triangulation of Δ_f dual to Γ_t is the same for each $t \in]0, \frac{1}{e}]$ and the triangulation of Δ_f dual to Γ_∞ is $\tau_{V_f, \mathcal{L}}$.

Our main aim in this section is to show that, the amoebas of the J_t -holomorphic hypersurfaces $H_t(V_{f_t})$'s cannot develop a new complement components in \mathbb{R}^n other than those of Γ_∞ . More precisely, we show that the spines of those amoebas have the same combinatorial type that the one of Γ_∞ .

Theorem 5.2. *Let $\mathcal{S} = \{t \in]0, \frac{1}{e}] \mid \text{the amoeba } \mathcal{A}_{H_t(V_{f_t})} \text{ is solid}\}$. Then \mathcal{S} is a nonempty, closed and open subset of $]0, \frac{1}{e}]$, and hence equal to $]0, \frac{1}{e}]$.*

Lemma 5.3. *For a sufficiently small t the amoebas $\mathcal{A}_{H_t(V_{f_t})}$ are solid. In particular \mathcal{S} is nonempty.*

Proof Assume that there exists an infinite sequence $\{t_m\}$ which tends to zero and such that the amoebas $\mathcal{A}_{H_{t_m}(V_{f_{t_m}})}$ are not solid, and the order of the complement component of the amoebas is $\beta \in \Delta_i$, where Δ_i is an element of the subdivision τ_∞ of Δ .

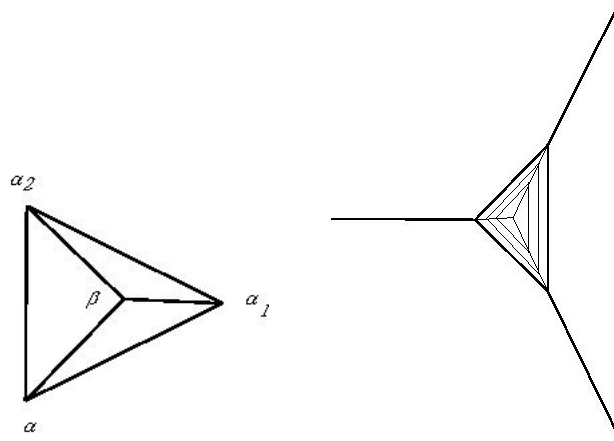
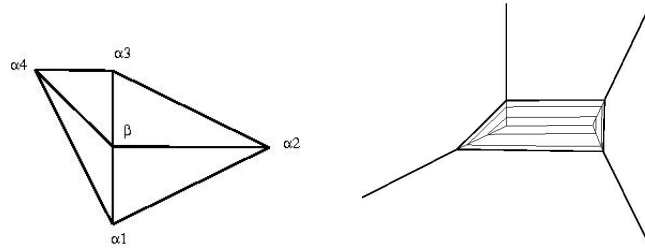
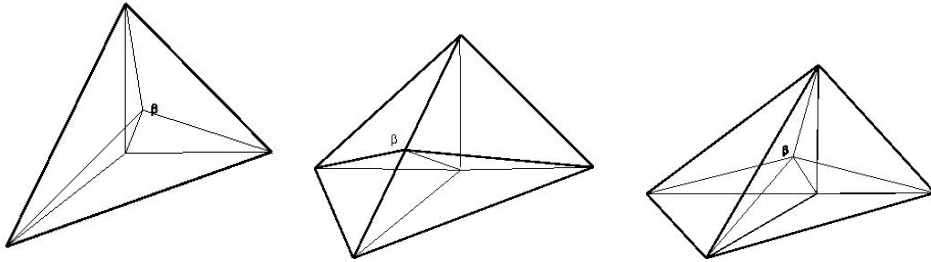
This means that there exists a sequence of parallel hyperplanes $\mathcal{P}_m \subset \mathbb{R}^n \times \mathbb{R}$ dual to the vertex β , such that $\mathcal{P}_m \cap \bigcup_{j=1}^l \mathcal{P}_{\alpha_j}$ is equal to the union of compact polyhedrons in Γ_{t_m} the spine of $\mathcal{A}_{H_{t_m}(V_{f_{t_m}})}$, and \mathcal{P}_{α_j} are the hyperplanes of $\mathbb{R}^n \times \mathbb{R}$ dual to the vertices α_j 's of Δ_i if $\beta \in \text{Int}(\Delta_i)$. The hyperplanes \mathcal{P}_{α_j} of $\mathbb{R}^n \times \mathbb{R}$ are the hyperplanes dual to the vertices α_j 's of $\Delta_i \cup \Delta_l$ if $\beta \in \Delta_i \cap \Delta_l$. Hence for any $n \geq 2$, the deformation can have n possibilities:

- (i) if the order β of the new complement component (i.e. of order not in $\text{Vert}(\Delta_f)$) is in the interior of some Δ_i we have the first possibility,
- (ii) and if the order of the new complement component is contained in a face of Δ_i we have $n - 1$ possibilities, one possibility for any positive dimension of the faces of Δ_i .

We can see the two possibilities when $n = 2$ in figures 7 and 8.

Let $\mathcal{P}_0 = \lim_{m \rightarrow \infty} \mathcal{P}_m$ which is a hyperplane parallel to the \mathcal{P}_m 's. If $\beta \in \text{Int}(\Delta_i)$, then either the \mathcal{P}_m 's passes through the lifting in $\mathbb{R}^n \times \mathbb{R}$ of the vertex v of Γ_∞ dual to Δ_i or they go to infinity. But if $\beta \in \partial(\Delta_i)$, then \mathcal{P}_0 is an hyperplane parallel to the \mathcal{P}_m 's and containing the dual of the sub-simplex of $\partial(\Delta_i)$ in which β lies, or \mathcal{P}_0 is parallel to the \mathcal{P}_m 's and goes to infinity. We can treat only the first case; the others can be given in the same way if we restrict ourself to the sub-simplex in the boundary of Δ_i containing β .

In other words, if we denote by $\nu_\infty(\beta)$ the limit of $\nu_m(\beta)$ when m tends to infinity (i.e. t_m tends to zero), with ν_m the Passare-Rullgård function corresponding to the

FIGURE 6. $n = 2$ and $\beta \in \text{Int}(\Delta_{\mathcal{C}_\alpha})$ FIGURE 7. $n = 2$ and $\beta \in \partial\Delta_{\mathcal{C}_\alpha}$ FIGURE 8. The three possibilities for $n = 3$

spine of the amoeba $\mathcal{A}_{f_{t_m}}$, we have a priori two possibilities :

$$\lim_{m \rightarrow \infty} \nu_m(\beta) = \begin{cases} -\infty & (a) \\ \text{or} \\ < \beta, a_v > +b_v & (b) \end{cases}$$

where $< \beta, a_v > +b_v$ is the equation of the hyperplane in $\mathbb{R}^n \times \mathbb{R}$ passing through the points of coordinates $(\alpha_j, \nu(\alpha_j))$ with $\alpha_j \in \text{Vert}(\Delta_i)$.

Let $g_t^{(m)}(z) = f_t(z) + t^{\nu_m(\beta)}(t - t_m)z^\beta$ and $V_{\infty, m}$ be the complex tropical hypersurface equal to the limit of $H_t(V_{g_t^{(m)}})$ when t tends to zero. Using theorem 5.1, it's clear that $\text{Log}(V_{\infty, m}) = \Gamma_{\infty, m}$ is the tropical hypersurface equal to the spine of the amoeba $\mathcal{A}_{H_{t_m}(V_{f_{t_m}})}$ and $\lim_{m \rightarrow \infty} \Gamma_{\infty, m} = \Gamma_\infty$.

Let us show that case (a) cannot occur. If $\lim_{m \rightarrow \infty} \nu_m(\beta) = -\infty$, then $\lim_{m \rightarrow \infty} c_{\beta, m} = \infty$ (the constants defined by Passare and Rullgård, see Section 2) and we have two cases:

- (i) either the hyperplane of \mathbb{R}^{n+1} corresponding to β intersect no other hyperplane corresponding to $\alpha \in \text{Vert}(\Delta)$, and then the monomial of index β is omitted from $g_t^{(m)}$ for each m ; and hence $\text{Log}^{-1}(v) \cap V_{\infty, f} = \text{Log}^{-1}(v) \cap V_{\infty, 0}$, with v the vertex in the non-Archimedean amoeba $\Gamma_\infty = \lim_{m \rightarrow \infty} \mathcal{A}_{H_{t_m}(V_{g_t^{(m)}})}$ dual to Δ_i , $V_{\infty, f}$ is the complex tropical hypersurface equal to the limit of $H_t(V_{f_t})$ when t tends to zero, and $V_{\infty, 0}$ is the limit of $V_{\infty, m}$ when m tends to infinity (with respect to the Hausdorff metric).
- (ii) or the amoeba Γ_∞ has a complement component of order β such that $c_{\beta, \infty} = +\infty$ and the coefficient $b_{\beta, m}$ of $g_t^{(m)}$ of index β evaluated at 0 is not bounded i.e. tends to ∞ . This case cannot occur because it contradicts the fact that the amoeba Γ_∞ is solid and nonempty.

In case (b), by multiplying f_t by t^p such that $p + b_v > 0$ if necessary, we can assume that $b_v > 0$. Recall that $V_{\infty, f} = \lim_{t \rightarrow 0} H_t(V_{f_t})$, $V_{\infty, m} = \lim_{t \rightarrow 0} H_t(V_{g_t^{(m)}})$ and $V_{\infty, 0} = \lim_{m \rightarrow \infty} V_{\infty, m}$.

Proposition 5.4. *In case (b) we have $V_{\infty, f} = V_{\infty, 0}$, in particular, if v is the vertex of Γ_∞ dual to the simplex Δ_i containing β , then $\text{Log}^{-1}(v) \cap V_{\infty, 0} = \text{Log}^{-1}(v) \cap V_{\infty, f}$.*

The problem is only on the 0-dimensional cell v of the non-Archimedean amoeba Γ_∞ dual to the simplex Δ_i of the triangulation and containing β . We denote by δ_l^k the k -dimensional cells of Γ_∞ containing v as vertex with $k = 1, \dots, n-1$, which are the dual to the $(n-k)$ -faces F_l^{n-k} of Δ_i of positive dimension. If τ_m is the triangulation of Δ dual to $\Gamma_{\infty, m}$, and $\Delta_l \subset \Delta_i$ are the elements of τ_m of maximal dimension, then we denote by $\delta_l^{(m), k}$ (resp. δ_l^k) the k -cells of $\Gamma_{\infty, m}$ (resp. of Γ_∞) which are dual to the $(n-k)$ -faces of the proper faces F_l^{n-1} of Δ_i for $k = 1, \dots, n-1$ (see M. Passare and H. Rullgård [PR1-04] for more details).

Lemma 5.5. *For all l and k we have :*

$$\lim_{m \rightarrow \infty} (V_{\infty, m} \cap \text{Log}^{-1}(\text{Int}(\delta_l^{(m), k}))) = V_{\infty, f} \cap \text{Log}^{-1}(\text{Int}(\delta_l^k)),$$

where $\text{Int}(\delta_l^{(m), k})$ is the interior of $\delta_l^{(m), k}$.

Proof Let $U(\delta_l^{(m),k}) \subset \mathbb{R}^n$ be a small open neighborhood of the interior of $\delta_l^{(m),k}$ satisfying the following properties: (i) its intersection with any other cells of Γ_m of dimension less than k is empty, and (ii) the limit of $U(\delta_l^{(m),k})$ when m tends to infinity is $\text{Int}(\delta_l^k)$ (see [V-90] page 42 for more details). Firstly we know that

$$V_{\infty,m} \cap \text{Log}^{-1}(\text{Int}(\delta_l^{(m),k})) = \lim_{t \rightarrow 0} H_t(V_{g_t^{(m)}}) \cap \text{Log}^{-1}(U(\delta_l^{(m),k})).$$

This means that for any m and any $\varepsilon > 0$, there exists $T_m < \frac{1}{\varepsilon}$ such that if $t \leq T_m$ then $H_t(V_{g_t^{(m)}}) \cap \text{Log}^{-1}(U(\delta_l^{(m),k}))$ is contained in an ε -neighborhood of $V_{\infty,m} \cap \text{Log}^{-1}(\text{Int}(\delta_l^{(m),k}))$. We look now at V_{f_t} as the end of a path γ of hypersurfaces in $(\mathbb{C}^*)^n$ starting at $V_{g_t^{(m)}}$, where the parameter of the path γ is the valuation of the coefficient of index β . This means that the coefficients of index different than β are independent of the parameter. By the continuity of roots property (see for example [B-71]) we have for any $\varepsilon > 0$ there exists $\eta > 0$ such that if $|\nu_m(b_\beta) - \nu_{m'}(b_\beta)| < \eta$ then the Hausdorff distance between $H_t(V_{g_t^{(m)}}) \cap \text{Log}^{-1}(U(\delta_l^{(m),k}))$ and $H_t(V_{f_t}) \cap \text{Log}^{-1}(U(\delta_l^{(m'),k}))$ is less than ε for a sufficiently small t (we can assume that there exists a very large m' such that $t = t_{m'}$). On the other hand $V_{\infty,f} \cap \text{Log}^{-1}(\text{Int}(\delta_l^k)) = \lim_{m \rightarrow \infty} H_{t_m}(V_{f_{t_m}}) \cap \text{Log}^{-1}(U(\delta_l^{(m),k}))$. By the triangular inequality of the Hausdorff distance, we obtain that for any $\varepsilon' > 0$, there exists m_1 such that if $m > m_1$ then we have

$$d_{\mathcal{H}}(V_{\infty,m} \cap \text{Log}^{-1}(\text{Int}(\delta_l^{(m),k})); V_{\infty,f} \cap \text{Log}^{-1}(\text{Int}(\delta_l^k))) < \varepsilon',$$

and the lemma is done. Proposition 5.4 is an immediate consequence of Lemma 5.5, and in particular, if v is the vertex of Γ_∞ dual to the simplex Δ_i containing β , then $\text{Log}^{-1}(v) \cap V_{\infty,0} = \text{Log}^{-1}(v) \cap V_{\infty,f}$.

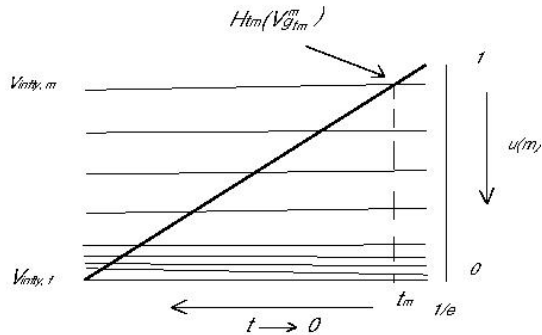


FIGURE 9.

■

End of proof of Lemma 5.3. The triangulation of Δ_f , dual to the spine Γ_{t_m} of the amoeba of the hypersurface $H_{t_m}(V_{g_t^{(m)}})$ is unchanged (i.e. the same for each m), because the Γ_{t_m} 's have the same combinatorial type. Indeed the set of slopes of faces of a tropical hypersurface is a finite set of rational numbers. Hence, by a restriction to a subsequence of $\{t_m\}$ if necessary, we may assume that the valuation of the coefficients $b_{\beta, m}$'s take their values in the interval $[-\nu_{m_0}(\beta), -\nu_{\infty}(\beta)]$ for some m_0 , so that there exists a strictly decreasing function $u : \{m \in \mathbb{N}/m > m_0\} \rightarrow [0, 1]$ with $\lim_{m \rightarrow \infty} u(m) = 0$ and $u(m_0) = 1$. So the valuation $\text{val}(a_{\beta, m}) = -\nu_m(\beta)$ can be written as follow : $(1 - u(m))(-\nu_{\infty}(\beta)) + u(m)(-\nu_{m_0}(\beta))$. So the case (b) satisfy the hypothesis (i) and (ii) which are the assumptions of Lemma 3.7.

By Proposition 4.1, if $U(v) \subset \mathbb{R}^n$ is a small ball centered on v , then the tropical localization says that for any $\varepsilon > 0$ there exist t_0 such that if $t \leq t_0$ then $\text{Arg}(H_t(V_{f_t}) \cap \text{Log}^{-1}(U(v)))$ is contained in the ε -neighborhood $\mathcal{N}_{\varepsilon}(\text{Arg}(V_{\infty, \Delta_i}))$ of the set of arguments of the complex tropical hypersurface V_{∞, Δ_i} (because the transformations H_t and $\Phi_{\Delta_i, t}$ conserve the arguments); this means that $\text{Arg}(V_{\infty, f} \cap \text{Log}^{-1}(v))$ is contained in an ε -neighborhood of $\text{Arg}(V_{\infty, \Delta_i})$, where $V_{\infty, \Delta_i} = \lim_{t \rightarrow 0} H_t(V_{f_t}^{\Delta_i})$.

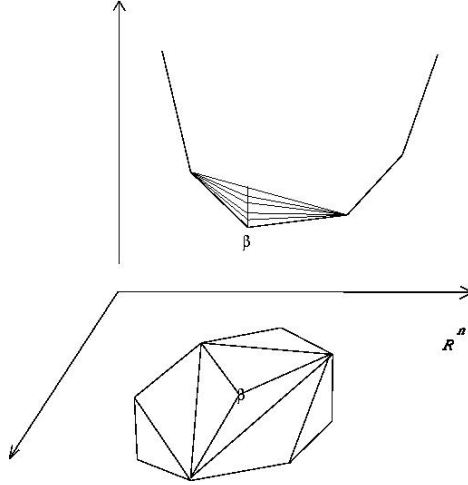


FIGURE 10. The deformation given by the valuation

On the other hand $\text{Arg}(H_t(V_{g_t^{(m, \Delta_i)}}) \cap \text{Log}^{-1}(U(v)))$ is contained in the ε -neighborhood $\mathcal{N}_{\varepsilon}(\text{Arg}(V'_{\infty, \Delta_i}))$ of the set of arguments of the complex tropical hypersurface V'_{∞, Δ_i} which is the limit of the complex tropical hypersurfaces $V_{\infty, g_m^{\Delta_i}}$ when m tends to ∞ , and $V_{\infty, g_m^{\Delta_i}} = \lim_{t \rightarrow 0} H_t(V_{g_t^{(m, \Delta_i)}})$ (here $g_t^{(m, \Delta_i)}$ is the truncation of $g_t^{(m)}$ to Δ_i), and then $\text{Arg}(V_{\infty, 0} \cap \text{Log}^{-1}(v))$ is contained in an ε -neighborhood of $\text{Arg}(V'_{\infty, \Delta_i})$. If \mathcal{P}_j is an extra-piece in $\text{Arg}(V'_{\infty, \Delta_i})$ (see Lemma 3.7), then we claim that $\mathcal{P}_j \cap \text{Arg}(V_{\infty, 0} \cap \text{Log}^{-1}(v))$ has a no vanishing volume. Indeed, assume that $\text{Vol}(\mathcal{P}_j \cap \text{Arg}(V_{\infty, 0} \cap \text{Log}^{-1}(v))) = 0$, hence there exists an external hyperplane H_{ij} for $\text{Arg}(V'_{\infty, \Delta_i})$

which is not external for $\text{Arg}(V_{\infty,0})$, such that $H_{ij} \cap \partial(\overline{\text{Arg}(V_{\infty,0})})$ is of dimension $n - 1$. But this situation cannot occur, because the hyperplane H_{ij} is not external for $\text{Arg}(V_{\infty,0})$.

The set of arguments $\text{Arg}(V_{\infty,f} \cap \text{Log}^{-1}(v))$ is contained in an ε -neighborhood of $\text{Arg}(V_{\infty,\Delta_{\mathcal{C}_\alpha}})$, and its intersection with \mathcal{P}_j is empty (recall that the polynomial f is maximally sparse, see Lemma 3.7). Hence $V_{\infty,f}$ and $V_{\infty,0}$ cannot be equal, and then we have a contradiction, which means that such sequence of t_m 's cannot exist. Hence for sufficiently small t , the amoebas $\mathcal{A}_{H_t(V_{f_t})}$ are solid and then the set \mathcal{S} is nonempty. ■

The following Corollary is a consequence of the last construction and Proposition 3.5.

Corollary 5.6. *Let $V_f \subset (\mathbb{C}^*)^n$ be an hypersurface defined by a maximally sparse polynomial f with Newton polytope a simplex. Then the amoeba of V_f is solid.*

Proof Assume instead that the amoeba \mathcal{A}_f is not solid; hence there exist $\beta \in \Delta \cap \mathbb{Z}^n$ which is the order of some complement component other than those corresponding to $\text{Vert}(\Delta)$. Let $g_t^{(m)}(z) = f_{t_m}(z) + e^{(1-u(m))s+u(m)\nu(\beta)} t^{(1-u(m))s+u(m)\nu(\beta)} (t_m - t)z^\beta$ where t_m is a sequence of real numbers which tends to zero, and $s = \langle \beta, a_v \rangle + b_v$, where $y = \langle x, a_v \rangle + b_v$ is the equation of the hyperplane in $\mathbb{R}^n \times \mathbb{R}$ containing the points of coordinates $(\alpha, \nu(\alpha))$ with $\alpha \in \text{Vert}(\Delta_f)$, and the sequence $u(m)$ is the sequence defined above. Using the above deformation and applying Proposition 3.5, we obtain that the complex tropical hypersurfaces $\text{Arg}(V_{\infty,f})$ and $\text{Arg}(V_{\infty,0})$ are different, because even if $s \gg \langle \beta, a_v \rangle + b_v$ and tends to infinity, the set of arguments $V_{\infty,0}$ contains extra-pieces corresponding to the no vanishing coefficients which have no contribution on the non-Archimedean amoeba. ■

Proof of theorem.5.2. By Lemma 5.3, the set \mathcal{S} is nonempty and it is obviously closed. Let t_{\max} be the maximum of $t \in \mathcal{S}$. We claim that t_{\max} is in the interior of the interval $]0; \frac{1}{e}]$ and then \mathcal{S} is open. Indeed, assume on the contrary that there is an infinite sequence $\{t_m\}$ in $]t_{\max}; \frac{1}{e}]$ such that $\lim_{m \rightarrow \infty} t_m = t_{\max}$, and the amoebas of the hypersurfaces $V_m = \{z \in (\mathbb{C}^*)^n \mid f_{t_m}(z) = 0\}$ are not solid. We know that the amoebas of the hypersurfaces defined by the truncated polynomials $f_t^{\Delta_i}$ are solid, because the Δ_i 's are a simplexes, and the set of its arguments contains no extra-pieces. For sufficiently large m , let we assume that f_{t_m} develop just one complement component of order β , and Δ_i is the element of τ containing β . Let $g_t^{(m)}$ be the family of polynomials defined by $g_t^{(m)}(z) = f_t(z) + t^{\nu(\beta)}(t - t_m)z^\beta$. Let us denote by $g_t^{(m;\Delta_i)}$ the truncation of $g_t^{(m)}$ to Δ_i , and $V_{(m;\Delta_i)}$ the complex tropical hypersurface which is the limit of $H_t(V_{g_t^{(m;\Delta_i)}})$ when t tends to zero (with respect to the Hausdorff metric on compacts). For any m , using the same reasoning as in Lemma 3.7, we show that the complex tropical hypersurface $V_{(\infty;\delta_i)}$ equal to the limit of $V_{(m;\Delta_i)}$ when

m tends to the infinity, contains extra-pieces with no vanishing volume. This means that the set of arguments of $H_{t_{max}}(V_{g_{t_{max}}(\infty; \Delta_i)})$ contains some extra-pieces. Contradiction because $H_{t_{max}}(V_{g_{t_{max}}(\infty; \Delta_i)})$ is equal to $H_{t_{max}}(V_{f_{t_{max}}^{\Delta_i}})$ (see Proposition 5.4). Hence there is no sequence $\{t_m\}$ such that $t_m \rightarrow t_{max}$ and the amoebas $\mathcal{A}_{H_{t_m}(V_{f_{t_m}})}$ are not solid, and then \mathcal{S} is open.

Corollary 5.7. *Let V be an algebraic hypersurface in $(\mathbb{C}^*)^n$ defined by a maximally sparse polynomial f with amoeba \mathcal{A}_V and Newton polytope Δ . Then the number of components of $\mathbb{R}^n \setminus \mathcal{A}_V$ is equal to the number of vertices of Δ .*

6. APPENDIX

A: Proposition 3.5 in the Case $n = 1$

Let us prove in this Appendix the Lemma 3.6 and proposition 3.5 in the case $n = 1$. Let $\Delta = [0, k]$, $A = \{\beta_1, \dots, \beta_s\} \subset]0, k[\cap \mathbb{Z}$. Firstly, we can remark that if $f_{\mathbb{K}}(z) = a_k z^k + a_0$, then we can assume that the coefficient a_k is equal to one and the valuation of the coefficient a_0 is zero. Indeed, the first assertion is obvious and for the



second we consider the translation Φ_{a_0} of \mathbb{K}^* defined by

$$\begin{aligned} \Phi_{a_0} : \mathbb{K}^* &\longrightarrow \mathbb{K}^* \\ z &\longmapsto t^{-\frac{\text{val}(a_0)}{k}} z \end{aligned}$$

and we obtain:

$$\begin{aligned} f_{\mathbb{K}} \circ \Phi_{a_0}(z) &= t^{-\text{val}(a_0)} z^k + a_0 \\ &= t^{-\text{val}(a_0)} (z^k + a'_0) \\ &= t^{-\text{val}(a_0)} f'_{\mathbb{K}}(z) \end{aligned}$$

with $\text{val}(a'_0) = 0$ and $\Phi_{a_0}^{-1}(V_{f_{\mathbb{K}}}) = V_{f'_{\mathbb{K}}}$. So we have:

$$W(V_{f_{\mathbb{K}}}) = e^{\frac{-\text{val}(a_0) + i \arg(a_0)}{k}} \cdot \{e^{i \frac{(2l+1)\pi}{k}}\}_{l=0}^{k-1}$$

Let $g_{\mathbb{K}, u}(z) = z^k + a_{\beta_1, u} z^{\beta_1} + \dots + a_0$ such that the initial part of $a_{\beta_1, u}$ is $t^{-\text{val}(a_{\beta_1, 1})u} e^{i \arg(a_{\beta_1, u})}$ and set

$g_{\mathbb{K}, u}^{[\beta_1, k]} = z^k + a_{\beta_1, u} z^{\beta_1}$ (the truncation of $g_{\mathbb{K}, u}$ to $[\beta_1, k]$). Therefore $W(V_{g_{\mathbb{K}, u}^{[\beta_1, k]}}) = e^{\frac{-\text{val}(a_{\beta_1, 1})u + i \arg(a_{\beta_1, 1})}{k - \beta_1}} \cdot \{e^{i \frac{(2l+1)\pi}{k - \beta_1}}\}_{l=0}^{k - \beta_1 - 1}$ and for each $j = 1, \dots, s$ we have in a similar way:

$$W(V_{g_{\mathbb{K}, u}^{[\beta_j, \beta_{j-1}]}}) = e^{A_j + i B_j} \cdot \{e^{i \frac{(2l+1)\pi}{\beta_{j-1} - \beta_j}}\}_{l=0}^{\beta_{j-1} - \beta_j - 1}$$

where $A_j = \frac{-\text{val}(a_{\beta_j,1})u + \text{val}(a_{\beta_{j-1},1})u}{\beta_{j-1} - \beta_j}$ and $B_j = \frac{\arg(a_{\beta_j,1}) - \arg(a_{\beta_{j-1},1})}{\beta_{j-1} - \beta_j}$. So we obtain:

$$W(V_{g_{\mathbb{K},0}}) = \bigcup_{j=1}^{s+1} e^{A_j + iB_j} \cdot \{e^{i \frac{(2l+1)\pi}{\beta_{j-1} - \beta_j}}\}_{l=0}^{\beta_{j-1} - \beta_j - 1}$$

with $a_{\beta_0,1} = a_k$, $a_{\beta_{s+1},1} = a_0$ and $\{\beta_j\}_{j=1}^s = A$. The first part of the lemma is done if we put $v_j = B_j$ for $j = 1, \dots, s+1$ and $L_j : x \mapsto (\beta_{j-1} - \beta_j)x$.

If $n = 1$ then we can see that $W(V_{g_{\mathbb{K},0}}) = W(V_{f_{\mathbb{K}}})$ if and only if $\beta_{j-1} - \beta_j$ is a constant r and $k = r(s+1)$. So it suffices to prove the assertion for the polynomials $f_{\mathbb{K}}^\circ(z) = z^{s+1} + a_0$ and $g_{\mathbb{K},u}^\circ(z) = z^{s+1} + a_0 + \sum_{j=1}^s a_{\beta_j,u} z^j$. By an easy computation we have $a_{\beta_1,0} \neq 0$, so in this case the roots r_j of the polynomial $g_{\mathbb{K},u}^\circ$ cannot have the same absolute value, because if it is the case then their sum is zero (because of the condition on their arguments). But in the case when the absolute value of the roots r_j are not the same then the amoeba of the limit of the $g_{\mathbb{K},u}^\circ$, when u tends to zero, have at least two points, which contradict the fact that the limit of the non-Archimedean amoeba has only one point.

B: The Set of Arguments of the Standard Hyperplane

let \mathcal{P}_{std} be the hyperplane in $(\mathbb{C}^*)^n$ defined by the polynomial $f(z_1, \dots, z_n) = 1 + \sum_{i=1}^n z_i$ with Newton polytope the standard simplex. If $(S^1)_l^{n-1}$ is the $(n-1)$ -torus in $(S^1)^n$ defined by: $\{(x_1, \dots, x_{l-1}, e^{i\pi}, x_{l+1}, \dots, x_n) \in (S^1)^n\}$ for $l = 1, \dots, n$, then the lift in \mathbb{R}^n of the union $\bigcup_{l=1}^n (S^1)_l^{n-1}$ divide a fundamental domain of the torus into 2^n parts $\{\tau_s\}_{s=1}^{2^n}$. Let τ_1 be the n -cube in \mathbb{R}^n of vertices $(v_1, \dots, v_{n-1}, \pi)$, $(v_1, \dots, v_{n-1}, 2\pi)$ with $v_j \in \{0, \pi\}$, and \mathcal{C}_1 is the cone of vertex $v_0 = (0, \dots, 0, \pi)$ and base the $(n-1)$ -cube of vertices $(v_1, \dots, v_{n-1}, 2\pi)$ with $v_j \in \{0, \pi\}$.

Lemma 6.1. *The image under the argument map Arg of the hyperplane $\mathcal{P}_{std} \subset (\mathbb{C}^*)^n$ defined by the equation $1 + \sum_{i=1}^n z_i = 0$ is the union of the $2^n - 2$ polyhedron $\mathcal{D}_s = \tau_s \setminus \mathcal{C}_s$ (not convex for $n > 2$) such that :*

- (a) *The τ_s are different than the two following cubes: (i) τ_0 of vertices $(v_1, \dots, v_{n-1}, \pi)$, $(v_1, \dots, v_{n-1}, 0)$ with $v_j \in \{0, \pi\}$, and (ii) τ_π of vertices $(v_1, \dots, v_{n-1}, \pi)$ and $(v_1, \dots, v_{n-1}, 2\pi)$ with $v_j \in \{\pi, 2\pi\}$,*
- (b) *the polyhedrons \mathcal{D}_s viewed as subset of the torus (by projection), are two by two attached by $2^n - 1$ real points of $(S^1)^n$, and the complement in $(S^1)^n$ of the closure of their union is a connected and convex polyhedron,*
- (c) *the volume of the coamoeba of \mathcal{P}_{std} is equal to $\frac{(n-1)(2^n-2)}{n} \pi^n$ (with respect to the flat metric of the torus).*

Proof If $n = 2$, then we have $z_2 = -(1 + z_1)$, so $\arg(z_2) = \pi + \arg(1 + z_1) \pmod{2\pi}$. Hence if $\arg(z_1) = \alpha < \pi$ and its module varies between zero and the infinity, then $\arg(z_2)$ varies between π and $\pi + \alpha$. By switching the role of the variable z_1 and z_2 ,

the lemma is done (see figure 8). For $n > 2$, we use induction on n and the fact that $z_n = -(1 + \sum_{j=1}^{n-1} z_j)$. Put $\alpha_j = \arg(z_j)$, so if $0 \leq \alpha_j \leq \pi$ for $j = 1, \dots, n-1$ then $\alpha_n \in]\pi, m_n[$ where $m_n = \max_{1 \leq j \leq n-1} \{\alpha_j\}$ and then we have one of the sets \mathcal{D}_s . By changing the position of the arguments of the z_j 's we obtain the $2^n - 2$ sets. Convexity is local property, which is the case in our situation, so the second statement of the lemma is obvious. For the third affirmation of the lemma, it suffice to compute the volume of the cone's \mathcal{C}_s which is equal to π^n/n and then the volume of any \mathcal{D}_s is $\pi^n - \pi^n/n = \frac{(n-1)}{n}\pi^n$. I leave the details to the reader.

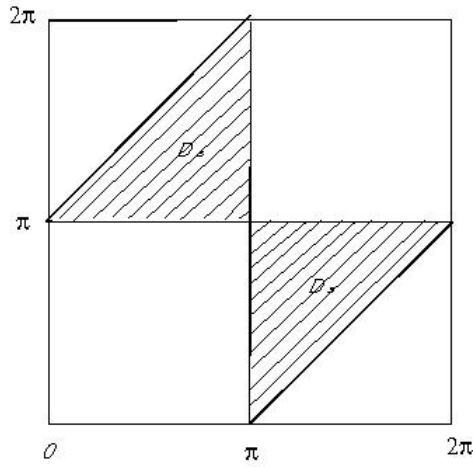


FIGURE 11. The left picture represent one $\mathcal{D}_s = \tau_s \setminus \mathcal{C}_s$ and the right one represent all the argument of a line which has two \mathcal{D}_s 's subsets.

■

Remark 6.2. We can see, using the result of section 3, that if $g \in \mathbb{K}[z_1, \dots, z_n]$ is a maximally sparse polynomial with Newton polytope a simplex Δ , and V_g is the hypersurface in $(\mathbb{K}^*)^n$ defined by g , then the volume of the set of arguments of $W(V_g)$ is equal to the volume of the set of argument of the standard hyperplane \mathcal{P}_{std} in

$(\mathbb{C}^*)^n$ defined by the polynomial $f(z_1, \dots, z_n) = 1 + \sum_{i=1}^n z_i$ (the torus is equipped with

the flat metric). Indeed, $\text{Arg}(W(V_g)) = {}^t L_{\Delta}^{-1} \text{Arg}(\mathcal{P}_{std})$ (viewed in some fundamental domain in \mathbb{R}^n the universal covering of the torus), where L_{Δ} is the linear part of the affine linear surjection $\rho : \Delta_{std} \rightarrow \Delta$. Firstly we have $\text{Vol}(\mathcal{D}_s) = \frac{\text{Vol}(\mathcal{D}'_s)}{\det(L_{\Delta})}$, where \mathcal{D}'_s are the polyhedrons in the torus $(S^1)^n$ corresponding to $W(V_g)$, and for any \mathcal{D}_s it corresponds $\det(L_{\Delta})$ times \mathcal{D}'_s . Hence we have $\text{Vol}(\text{Arg}(W(V_g))) = \text{Vol}(\cup \mathcal{D}'_s) = \det(L_{\Delta}) \left(\frac{\cup \mathcal{D}_s}{\det(L_{\Delta})} \right) = \text{Vol}(\text{Arg}(\mathcal{P}_{std}))$.

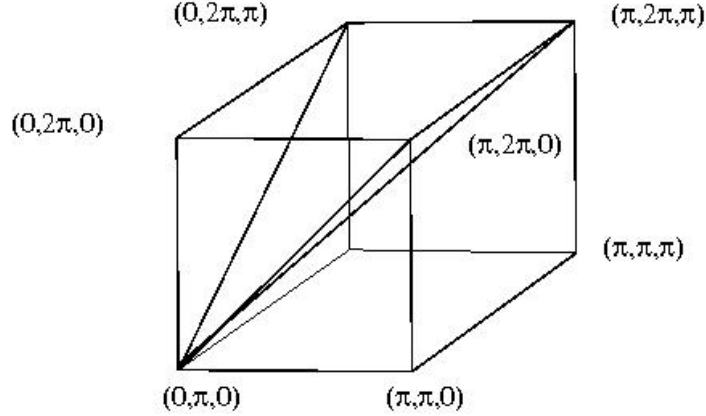


FIGURE 12. If $n = 3$, we represent here one $\mathcal{D}_s = \tau_s \setminus \mathcal{C}_s$ where τ_s is all the cube, and \mathcal{C}_s is the cone of base the square in the top and vertex the point $(0, \pi, 0)$.

C: Construction of the Triangulation $\tau_{V_f, \mathcal{L}}$

We use the notation of Section 5 where $\mathcal{L} \subset \Delta_f \cap \mathbb{Z}^n$ denotes the complement of $\text{Vert}(\Delta_f)$ in the image of the order mapping i.e. $\mathcal{L} = \{\alpha \in \Delta_f \cap \mathbb{Z}^n \mid \text{with } E_\alpha^c \neq \emptyset \text{ and } a_\alpha = 0\}$; the polynomial f is assumed maximally sparse. Using the triangulation τ_{V_f} dual to the spine of the amoeba \mathcal{A}_{V_f} , we define a new triangulation $\tau_{V_f, \mathcal{L}}$ of Δ_f as follow:

Step.1 Let $\alpha_1 \in \mathcal{L}$ and denote by $\Delta_{\mathcal{C}_{\alpha_1}^j}$ the following subsets of τ_{V_f} :

- (1) $\Delta_{\mathcal{C}_{\alpha_1}^1} = \cup_i \Delta_i$ is a convex subset of Δ_f containing α_1 where Δ_i 's are element of τ_{V_f} ,
- (2) $\text{Vert}(\Delta_{\mathcal{C}_{\alpha_1}^1}) \subseteq \text{Vert}(\Delta_f)$,
- (3) for any proper face $F_{\Delta_{\mathcal{C}_{\alpha_1}^1}}$ of $\Delta_{\mathcal{C}_{\alpha_1}^1}$ we have $\partial F_{\Delta_{\mathcal{C}_{\alpha_1}^1}} \subset \cup_j \partial F_{\Delta_f}^j$ where $\cup_j F_{\Delta_f}^j = \partial \Delta_f$
- (4) $\Delta_{\mathcal{C}_{\alpha_1}^1}$ is of minimal volume with properties (1), (2) and (3).

If there exist r subsets of τ_{V_f} satisfying (1), (2), (3) and (4) with $r \geq 3$ and the interior of $\Delta_{\mathcal{C}_{\alpha_1}^i} \cap \Delta_{\mathcal{C}_{\alpha_1}^j}$ is empty, then we associate to α_1 the union $\cup_{j=1}^r \Delta_{\mathcal{C}_{\alpha_1}^j}$. We can remark that this case can occur only if α_1 is contained in the boundary of a proper face of Δ_f .

If α_1 is in the interior of a proper face of $\Delta_{\mathcal{C}_{\alpha_1}^1}$, then either there is many other $\Delta_{\mathcal{C}_{\alpha_1}^j}$'s satisfying (1),(2),(3) and (4), in this case we associate to α_1 their union, or there is only one $\Delta_{\mathcal{C}_{\alpha_1}^1}$ satisfying (1),(2),(3) and (4) and then we take the connected component of $\Delta_f \setminus \Delta_{\mathcal{C}_{\alpha_1}^1}$ containing α_1 and we repeat the same operation on this component. So we obtain $\Delta_{\mathcal{C}_{\alpha_1}^j}$'s simplex with

$\text{Vol}(\Delta_{\mathcal{C}_{\alpha_1}^j}) \geq \text{Vol}(\Delta_{\mathcal{C}_{\alpha_1}^1})$. As in the second case we associate to α_1 the union of $\Delta_{\mathcal{C}_{\alpha_1}^1}$ with the $\Delta_{\mathcal{C}_{\alpha_1}^j}$'s. If $\alpha_1 \in \text{Int}(\Delta_{\mathcal{C}_{\alpha_1}^1})$ then there is a unique subset $\Delta_{\mathcal{C}_{\alpha_1}^1}$ of τ_{V_f} satisfying (1), (2), (3) and (4).

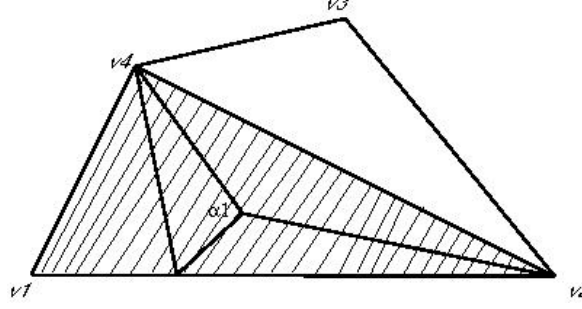


FIGURE 13. The dashed triangle represent one $\Delta_{\mathcal{C}_{\alpha_1}^1}$

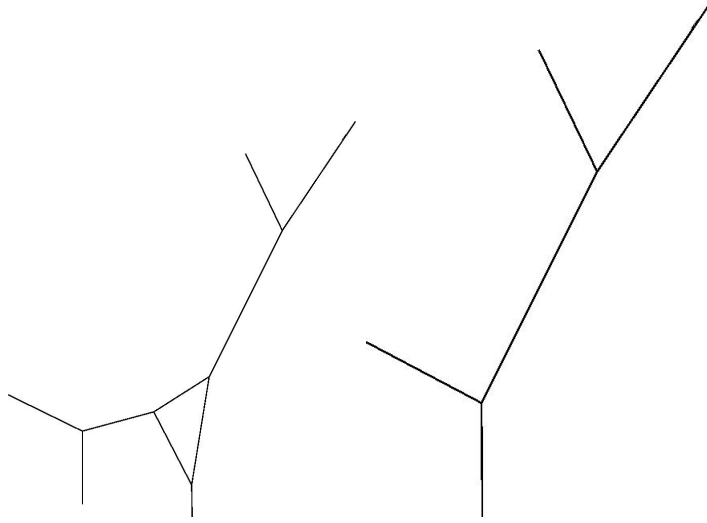


FIGURE 14. The spines Γ_t have the same combinatorial type of the left picture and Γ_0 has the combinatorial type of the right one

Step.2 Let $K_{\alpha_1} = \Delta_f$ and K_{α_2} be the connected component of $\Delta_f \setminus (\cup_j \Delta_{\mathcal{C}_{\alpha_1}^j})$ containing α_2 and we repeat the same operation for α_2 . By this process we obtain a new subdivision $\tau_{V_f, \mathcal{L}}$ of Δ_f such that $\tau_{V_f, \mathcal{L}} = \mathcal{P} \cup \mathcal{R}$ where $\mathcal{P} = \cup_{i=1}^s \cup_{j=1}^{r_i} \Delta_{\mathcal{C}_{\alpha_i}^j}$ and \mathcal{R} is the union of element in τ_{V_f} not in \mathcal{P} .

Let $\Gamma_{\mathcal{C}_{\alpha_i, t}^j}$ be the tropical hypersurface dual to $\tau_{V_f|\Delta_{\mathcal{C}_{\alpha_i}^j}}$ such that all its vertices are in $\text{Vert}(\Gamma_t)$ where Γ_t is the spine of the amoeba $\mathcal{A}_{H_t(V_{f_t})}$. We denote by $\mathcal{C}_{\alpha_i, t}^j$ the set of its bounded polyhedrons of dimension $n - 1$. If $\alpha \in \mathcal{L}$ we denote, in the sequel, by $\mathcal{C}_{\alpha, t}$ one of the $\mathcal{C}_{\alpha, t}^j$'s.

Definition 6.3. We said that a family of a polyhedron $P_t \subset \mathbb{R}^n$ vanishes or collapses if we have the following:

- (i) for each $t > 0$, the polyhedrons P_t are homothetic,
- (ii) their volume tends to zero when $t \rightarrow 0$.

This means that the dual of P_t is constant for each $t > 0$ and P_t collapses to some point.

Lemma 6.4. *Let $\alpha \in \mathcal{L}$. Then the set $\mathcal{C}_{\alpha, t}$ vanishes.*

Proof If the amoebas $\mathcal{A}_{H_t(V_{f_t})}$ converge (with respect to the Hausdorff metric in the compact subsets of \mathbb{R}^n) to a tropical hypersurface Γ_∞ , then also, their spines converge to Γ_∞ . In particular the number of polyhedrons in Γ_∞ of maximal dimension (i.e. $n - 1$) is not greater than the number of polyhedrons of Γ_t of maximal dimension. This means that some polyhedrons $P_t \subset \Gamma_t$ converge to a parallel polyhedron $P \subset \Gamma_\infty$ (because the set of slopes of faces of a tropical hypersurface is a finite set of rational numbers) and some other one vanished.

Let $\alpha \in \mathcal{L}$ and $\Gamma_{\mathcal{C}_{\alpha, t}}$ be the dual of $\tau_{V_f|\Delta_{\mathcal{C}_\alpha}}$ for $\alpha \in \mathcal{L}$. By definition, the set of vertices of $\Gamma_{\mathcal{C}_{\alpha, t}}$ is contained in $\text{Vert}(\Gamma_t)$, on the other hand, if K is a compact in \mathbb{R}^n containing all vertices of $\Gamma_{\mathcal{C}_{\alpha, t}}$, then $\Gamma_t \cap \Gamma_{\mathcal{C}_{\alpha, t}}$ converges to the intersection of K with the tropical hypersurface $\Gamma_{\Delta_{\mathcal{C}_\alpha}}$ dual to $\Delta_{\mathcal{C}_\alpha}$ (we mean here the dual to the subdivision of the polytope $\Delta_{\mathcal{C}_\alpha}$ with only one element). Hence $\mathcal{C}_{\alpha, t}$ vanishes, because $\Gamma_{\Delta_{\mathcal{C}_\alpha}}$ has only one vertex, and then it has no compact subpolyhedrons other than its vertex. ■

Remark 6.5. We can remark that the subdivision $\tau_{V_f, \mathcal{L}}$ is a convex subdivision of Δ_f and defined by the Passare-Rullgård's function ν restricted to the set of vertices of Δ . So $\tau_{V_f, \mathcal{L}}$ is dual to Γ_∞ and hence $\tau_{V_f, \mathcal{L}} = \tau_\infty$.

D: Maximally sparse is an optimal condition

We give in this Appendix an example of a curve $V_f \subset (\mathbb{C}^*)^2$ defined by a polynomial f of Newton polygon Δ such that its support contains other elements than those of the vertices of Δ , and the number of the complement components of the amoeba \mathcal{A}_f is strictly greater than the cardinality of the support of f . Let V_f be the curve in $(\mathbb{C}^*)^2$ defined by the polynomial $f(z, w) = -zw^2 + z^3w - 7zw + 6w + z$. We can see, by computation that the points $(0, 0)$, $(\log(2), 0)$ and $(\log(3), 0)$ are contained in the amoeba \mathcal{A}_f of the curve V_f and the two points $(\frac{\log(2)}{2}, 0)$ and $(\frac{\log(3)}{2}, 0)$ are contained in two compact different complement components of \mathcal{A}_f . Hence the number

of complement components in \mathbb{R}^2 of \mathcal{A}_f is equal to 6 which is strictly greater than the number of monomials of f .

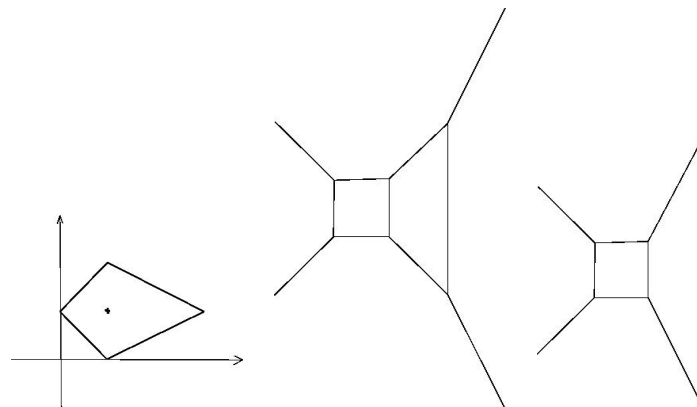


FIGURE 15. The Newton polygon of f , the spine of \mathcal{A}_f , and the non-Archimedean amoeba Γ_∞

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